1 Perfect Square

Note 2

(a) Prove that if n^2 is odd, then n must also be odd.

(b) Prove that if n^2 is odd, then n^2 can be written in the form $8k+1$ for some integer k.

Solution:

(a) We will proceed by a proof by contraposition; the contrapositive of the statement is "if n is even, then n^2 is also even". Here, since n is even, we can write $n = 2k$ for some integer k. This makes $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even, as desired. By contraposition, this means that if n^2 is odd, then n must also be odd.

(b) We will proceed with a direct proof. From the previous part, since n^2 is odd, n is also odd, i.e., of the form $n = 2l+1$ for some integer l. Then, $n^2 = 4l^2 + 4l + 1 = 4l(l+1) + 1$. Since one of l and $l+1$ must be even, $l(l+1)$ is of the form $2k$ for some integer k and $n^2 = 8k+1$.

2 Numbers of Friends

Note 2

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.

(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \ldots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), is friends with a person who has 0 friends (i.e., friends with no one). This is a contradiction since friendship is mutual.

Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to n possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0, 1, ..., n-1$. The objects assigned to these containers are the people at the party. However, containers 0, $n-1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning n people to at most $n-1$ containers, and by the pigeonhole principle, at least one of the $n-1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.

3 Pebbles

Note 2

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones.

Prove that there must exist an all-red column.

Solution: We give a proof by contraposition; the contrapositive is "if there does not exist an all-red column, then there is always a way of choosing one pebble from each column such that there does not exist a red pebble among the chosen ones".

Suppose there does not exist an all-red column. This means that we can always find a blue pebble in each column. Therefore, if we take one blue pebble from each column, we have a way of choosing one pebble from each column without any red pebbles. This is the negation of the original hypothesis, so we are done.

We can also approach the problem through contradiction; the logic stays almost exactly the same, and we start with the negation of the conclusion: that there does not exist an all-red column. The same reasoning above allows us to conclude that there will always exist a way of choosing one pebble from each column such that all pebbles are blue (i.e. no pebbles are red).