1 Counting Intro I

Note 10

Zeroth rule of counting: For sets A, B if there exists a bijection $f : A \to B$, then $|A| = |B|$.

First rule of counting: When counting the number of ways to count a sequences of k choices, if there are n_1 options for the first choice, n_2 options for the second choice regardless of the decision in the first choice, etc., then there are a total of $n_1 \cdot n_2 \cdots n_k$ ways to make all k choices.

Second rule of counting: Let B be the set of unordered objects for counting a sequence of choices, and let A be the ordered set of such objects. If there exists an m-to-1 mapping $f : A \to B$, then $|A| = m \cdot |B|$.

In this problem, we will derive the formulas for rerranging k items out of n total:

	with replacement	without replacement
order matters	n^k	$\frac{n!}{(n-k)!}$
order doesn't matter	$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$	$\frac{n!}{k!(n-k)!} = \binom{n}{k}$

- Order matters, without replacement: also known as *permutations*
- Order doesn't matter, without replacement: also known as *combinations*
- Order doesn't matter, with replacement: also known as *stars and bars*. Here, stars and bars is usually used to compute the number of ways to split k items into n categories (hence k stars, and $n-1$ bars to separate the n categories).

Most counting problems involve a mix of all 4 methods; don't box yourself into using just one of these!

In the below questions, do not blindly apply the above formulas—try to re-derive them yourself!

(a) How many 4-card hands are there in a standard 52-card deck? (Any rearrangement of the cards in your hand is counted as identical.)

(b) A US phone number consists of a country code (always "+1"), an area code (any 3 digits), a telephone prefix (any 3 digits), and a line number (any 4 digits). How many US phone numbers are there?

(c) How many anagrams of the word "COUNT" are there? What about "BERKELEY"? (An *anagram* of a word is a rearrangement of its letters; for example, "CTONU" and "TNUOC" are both anagrams of "COUNT")

(d) The concept of "stars and bars" seems at first glance quite different from the idea of "order doesn't matter, with replacement". We'll look at how these two concepts are actually the same.

Consider the problem of counting the number of ways of splitting ten $1 bills among 4 people. How many ways of splitting the bills are there?

Compare this with the problem of counting the number of ways of arranging 10 numbers from the set $\{1,2,3,4\}$, where order doesn't matter and we have replacement. What is the relationship between the ways of counting these two scenarios? (*Hint*: Let each person be labeled by a digit from $\{1,2,3,4\}$. Can you represent a split of the ten bills with a sequence of digits?)

2 Inclusion and Exclusion

Note 10

What is the total number of positive integers strictly less than 100 that are also coprime to 100?

3 Farmer's Market

Note 10

Suppose you want k items from the farmer's market. Count how many ways you can do this, assuming:

(a) There are pumpkins and apples at the market.

(b) There are pumpkins, apples, oranges, and pears at the market.

(c) There are n kinds of fruits at the market, and you want to end up with at least two different types of fruit.

4 The Count

Note 10

(a) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

(b) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

(c) The Count now wants to make a password to secure his phone. His password must be exactly 10 digits long and can only contain the digits 0 and 1. On top of that, he also wants it to contain at least five consecutive 0's. How many possible passwords can he make?