# 1 Graph Theory Basics

Note 5

For each type of graph, select all properties that are satisfied by the graph type.

(a) Planar graph. Let $v$, $e$, $f$ be the number of vertices, edges, and faces, respectively.

i. $v+f=e+2$

ii. $e \leq 2v-4$

iii. Any planar graph can be vertex-colored in at most 5 colors.

(b) Tree

i. Has $|V|-1$ edges

ii. Can have cycles

iii. Adding an edge will decrease the number of connected components

(c) Hypercube. Let $n$ be the dimension of the hypercube.

i. $|V|=2^n$

ii. $|E|=n \cdot 2^{n-1}$

**Solution:**

(a) iii only. For i, Euler's formula requires that the graph is connected, so the formula does not hold for unconnected planar graphs. For ii, it holds true when the planar graph is bipartite. Note that for iii, it can be actually done with at most 4 colors.

(b) i only. For ii, a tree is always acyclic. For iii, adding an edge to a tree will not decrease the number of connected component since a tree is a connected graph, i.e., the number of connected component is 1.

(c) i, ii. ii can be derived from the handshake lemma since there are $2^n$ vertices in an $n$-dimensional hypercube, and each vertex has degree $n$, thus, $|E|=\frac{n \cdot 2^n}{2}=n \cdot 2^{n-1}$.

## 2 Always, Sometimes, or Never

Note 5

In each part below, you are given some information about a graph $G$. Using only the information in the current part, say whether $G$ will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) $G$ can be vertex-colored with 4 colors.

(b) $G$ requires 7 colors to be vertex-colored.

(c) $e \leq 3v - 6$, where $e$ is the number of edges of $G$ and $v$ is the number of vertices of $G$.

(d) $G$ is connected, and each vertex in $G$ has degree at most 2.

(e) Each vertex in $G$ has degree at most 2.

**Solution:**

(a) Either planar or non-planar. By the 4-color theorem, any planar graph can provide the planar example. The easiest non-planar example is $K_{3,3}$, which can be 2-colored because it is bipartite. (Certainly, any graph which can be colored using only 2 colors can also be colored using 4 colors.)

(b) Always non-planar. The 4-color theorem tells us that if a graph is planar, it can be colored using only 4 colors. The contrapositive of this is that if a graph requires more than 4 colors to vertex-color, it must be non-planar. (Using the 5- or 6-color theorem would also work.)

(c) Either planar or non-planar. From the notes, we know that every planar graph follows this formula, so any planar graph is a valid planar example. The easiest non-planar example is again $K_{3,3}$, which has $e = 9$ and $v = 6$, meaning our formula becomes $9 \leq 3(6) - 6 = 12$, which is certainly true.

(d) Always planar. There are two cases to deal with here: either $G$ is a tree, or $G$ is not a tree and so contains at least one cycle. In the former case, we're immediately done, since all trees are planar. In the latter case, consider any cycle in $G$. We know that every vertex in that cycle is adjacent to the vertex to its left in the cycle and to the vertex to its right in the cycle. But we also know that no vertex can be connected to more than two other vertices, so the cycle isn't connected to anything else. But $G$ is a connected graph, so we must have that $G$ is just a single large cycle. And we can certainly draw a simple cycle on a plane without crossing any edges, so even in this case $G$ is still planar.

Alternatively, we can use Kuratowski's theorem; since each vertex has a degree of at most 2, it is impossible for $G$ to contain $K_5$ or $K_{3,3}$. This means that $G$ must be planar.

(e) Always planar. Each of $G$'s connected components is connected and has no vertex of degree more than 2, so by the previous part, each of them must be planar. Thus, each of $G$'s connected components must be planar, so $G$ itself must be planar.

Alternatively, we can follow the same procedure as the previous alternate solution; each vertex still has a degree of at most 2, so it is impossible for $G$ to contain $K_5$ or $K_{3,3}$. This means that $G$ must be planar.

## 3 Graph Coloring

Note 5 Prove that a graph with maximum degree at most $k$ is $(k+1)$-colorable.

**Solution:**

The natural way to try to prove this theorem is to use induction on the graph's maximum degree, $k$. Unfortunately, this approach is extremely difficult because covering all possible types of graphs when maximum degree changes requires extreme caution. You might be envisioning a certain graph as you write your proof, but your argument will likely not generalize. In graphs, typical good choices for the induction parameter are $n$, the number of nodes, or $e$, the number of edges. We typically shy away from inducting on degree.

We use induction on the number of vertices in the graph, which we denote by $n$. Let $P(n)$ be the proposition that an $n$-vertex graph with maximum degree at most $k$ is $(k+1)$-colorable.

*Base Case $n = 1$:* A 1-vertex graph has maximum degree 0 and is 1-colorable, so $P(1)$ is true.

*Inductive Step:* Now assume that $P(n)$ is true, and let $G$ be an $(n+1)$-vertex graph with maximum degree at most $k$. Remove a vertex $v$ (and all edges incident to it), leaving an $n$-vertex subgraph,$H$. The maximum degree of $H$ is at most $k$, and so $H$ is $(k+1)$-colorable by our assumption $P(n)$. Now add back vertex $v$. We can assign $v$ a color (from the set of $k+1$ colors) that is different from all its adjacent vertices, since there are at most $k$ vertices adjacent to $v$ and so at least one of the $k+1$ colors is still available. Therefore, $G$ is $(k+1)$-colorable. This completes the inductive step, and the theorem follows by induction.

## 4 Hypercubes

Note 5 The vertex set of the $n$-dimensional hypercube $G = (V,E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all $n$-bit strings). There is an edge between two vertices $x$ and $y$ if and only if $x$ and $y$ differ in exactly one bit position.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that the edges of an $n$-dimensional hypercube can be colored using $n$ colors so that no pair of edges sharing a common vertex have the same color.

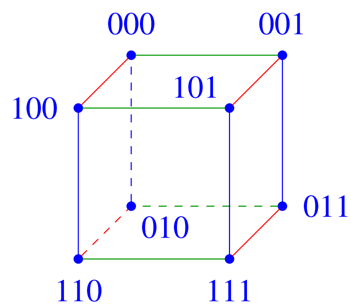
(c) Show that for any $n \geq 1$, the $n$-dimensional hypercube is bipartite.

**Solution:**

(a) The three hypercubes are a line, a square, and a cube, respectively. See also note 5 for pictures.

(b) Consider each edge that changes the $i$th bit for some $i \leq n$. Every vertex touches exactly one of these edges, because there is exactly one way to change the $i$th bit in any bitstring. Coloring each of these edges color $i$ ensures that each vertex will then be adjacent to $n$ differently colored edges, since there are $n$ different bits to change, and no two edges representing bit changes on different bits have the same color.

An example for the three dimensional case is shown below (red is the first bit, blue is the second bit, and green is the third bit):



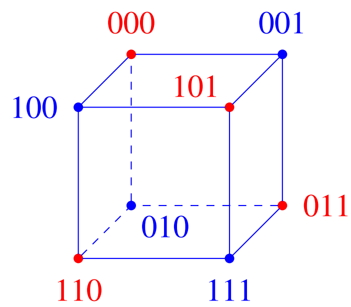

Alternate solution (using induction):

In the base case of $n = 1$, the hypercube of only one line can be edge colored with 1 color. Next, suppose that the $n$ dimensional hypercube can be colored with $n$ colors. Recall that the $n+1$ dimensional hypercube is composed of two $n$ dimensional hypercubes; each of these hypercubes can be colored with $n$ colors by the inductive hypothesis.

We can connect the two $n$ dimensional hypercubes with edges colored with a different color; this will be our $(n+1)$th color. Since these new edges will always be between distinct pairs of vertices, one from each subcube, none of these new edges will share a vertex, giving a valid coloring of the $n+1$ dimensional hypercube with $n+1$ colors.

(c) Consider the vertices with an even number of 0 bits and the vertices with an odd number of 0 bits. Each vertex with an even number of 0 bits is adjacent only to vertices with an odd number of 0 bits, since each edge represents a single bit change (either a 0 bit is added by flipping a 1 bit, or a 0 bit is removed by flipping a 0 bit). Let $L$ be the set of the vertices with an even number of 0 bits and let $R$ be the vertices with an odd number of 0 bits, then no two adjacent vertices will belong to the same set.

An example for the three dimensional case is shown below ($L$ are blue vertices, and $R$ are red vertices):




Alternate solution (using induction and coloring):

It may be simpler to that a graph being 2-colorable is the same as being bipartite. Now, the argument is easier to state. First the base case is a hypercube with two vertices which is clearly two-colorable. Then notice, switching the colors in a two-coloring is still valid as if endpoints are differently colored, switching leaves them differently colored. Now, recursively one two colors the two subcubes the same, and then switches the colors in one subcube. The internal to subcube edges are fine by induction. The edges across are fine as the corresponding vertices are differently colored due to the switching.