CS 70 Discrete Mathematics and Probability Theory
Summer 2025 Tate **DIS 3C**

# 1 Countability Intro

Note 11

A function $f : A \to B$ maps elements from set $A$ to set $B$.

$f$ is *injective* if it maps distinct elements to distinct elements, and *surjective* if every element in $B$ is mapped to by some element in $A$. If $f$ is both injective and surjective, it is *bijective*, and the sets $A$ and $B$ are said to have the same *cardinality* (size). The cardinality of a set is denoted by $|A|$.

$f$ is bijective if and only if there exists an inverse function $f^{-1} : B \to A$ such that $f^{-1}(f(a)) = a$ for all $a \in A$ and $f(f^{-1}(b)) = b$ for all $b \in B$.

**Countability**: Formal notion of different kinds of infinities.

- *Countable*: able to enumerate in a list (possibly finite, possibly infinite)
- *Countably infinite*: able to enumerate in an infinite list; that is, there is a bijection with $\mathbb{N}$.

  To show that there is a bijection, the *Cantor–Bernstein theorem* says that it is sufficient to find two injections, $f : S \to \mathbb{N}$ and $g : \mathbb{N} \to S$. Intuitively, this is because an injection $f : S \to \mathbb{N}$ means $|S| \leq |\mathbb{N}|$, and an injection $g : \mathbb{N} \to S$ means $|\mathbb{N}| \leq |S|$; together, we have $|\mathbb{N}| = |S|$.

- *Uncountably infinite*: unable to be listed out

  Use *Cantor diagonalization* to prove uncountability through contradiction; the classic example is the set of reals in $[0,1]$:

| $\mathbb{N}$ | $[0,1]$ |
|---|---|
| 0 | 0 . ⑦ 3 2 0 5 0 ··· |
| 1 | 0 . 4 ① 4 2 1 3 ··· |
| 2 | 0 . 6 1 ⑧ 0 3 3 ··· |
| 3 | 0 . 1 8 2 ⑧ 1 8 ··· |
| 4 | 0 . 1 4 1 5 ⑨ 2 ··· |
| 5 | 0 . 5 7 7 2 1 ⑤ ··· |
| ⋮ | ⋮ |
| ? | 0 . 8 2 9 9 1 6 ··· |

  If we change the digits along the diagonal, the new decimal created is different from every single element in the list in at least one place, so it's not in the list—this is a contradiction.

Sometimes it can be easier to prove countability/uncountability through bijections with other countable/uncountable sets respectively. Common countable sets include $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N} \times \mathbb{N}$, finite length bitstrings, etc. Common uncountable sets include $[0,1]$, $\mathbb{R}$, infinite length bitstrings, etc.

(a) Your friend is confused about how Cantor diagonalization doesn't apply to the set of natural numbers. They argue that natural numbers can be thought of as an infinite length string of digits, by padding each number with an infinite number of zeroes to the left. If we then assume by contradiction that we can list out the set of natural numbers with the padded 0's, we can change the digits along a diagonal, to create a new natural number not in the list.

$$\begin{array}{c|cccccc}
0 & \cdots & 0 & 0 & 0 & 0 & \textcircled{1} \\
1 & \cdots & 0 & 1 & 2 & \textcircled{3} & 4 \\
2 & \cdots & 5 & 2 & \textcircled{8} & 2 & 3 \\
3 & \cdots & 9 & \textcircled{4} & 3 & 2 & 1 \\
\vdots & & & & \vdots & & \\
\hline
? & \cdots & 1 & 5 & 9 & 4 & 2
\end{array}$$

What is wrong with this argument?

**Solution:**

(a) The issue here is that the newly created number is not necessarily a natural number. This number has an infinite number of digits (we'll always be changing the padded zeroes into some nonzero digits), while natural numbers must have a finite number of digits.

This means that when you perform Cantor diagonalization to show a set $S$ is uncountable, it is imperative that you ensure that the newly created element is always still an element of $S$; otherwise, we cannot say anything about the result of the diagonalization.

## 2 Count It!

Note 11 For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) The integers which divide 8.

(b) The integers which 8 divides.

(c) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(d) The set of strings over the English alphabet. (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.)

(e) The set of finite-length strings drawn from a countably infinite alphabet, $\mathscr{A}$.

(f) The set of infinite-length strings over the English alphabet.

**Solution:**

(a) Finite. They are $\{-8,-4,-2,-1,1,2,4,8\}$.

(b) Countably infinite. We know that there exists a bijective function $f : \mathbb{N} \to \mathbb{Z}$. Then the function $g(n) = 8f(n)$ is a bijective mapping from $\mathbb{N}$ to integers which 8 divides.

(c) Uncountably infinite. We use Cantor's Diagonalization Proof:

Let $\mathscr{F}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. We can represent a function $f \in \mathscr{F}$ as an infinite sequence $(f(0), f(1), \cdots)$, where the $i$-th element is $f(i)$. Suppose towards a contradiction that there is a bijection from $\mathbb{N}$ to $\mathscr{F}$:

$$\begin{aligned}
0 &\longleftrightarrow (f_0(0), f_0(1), f_0(2), f_0(3), \dots) \\
1 &\longleftrightarrow (f_1(0), f_1(1), f_1(2), f_1(3), \dots) \\
2 &\longleftrightarrow (f_2(0), f_2(1), f_2(2), f_2(3), \dots) \\
3 &\longleftrightarrow (f_3(0), f_3(1), f_3(2), f_3(3), \dots) \\
&\vdots
\end{aligned}$$

Consider the function $g : \mathbb{N} \to \mathbb{N}$ where $g(i) = f_i(i) + 1$ for all $i \in \mathbb{N}$. We claim that the function $g$ is not in our finite list of functions. Suppose for contradiction that it were, and that it was the $n$-th function $f_n(\cdot)$ in the list, i.e., $g(\cdot) = f_n(\cdot)$. However, $f_n(\cdot)$ and $g(\cdot)$ differ in the $n$-th argument, i.e. $f_n(n) \neq g(n)$, because by our construction $g(n) = f_n(n) + 1$. Contradiction!

(d) Countably infinite. The English language has a finite alphabet (52 characters if you count only lower-case and upper-case letters, or more if you count special symbols – either way, the alphabet is finite).

We will now enumerate the strings in such a way that each string appears exactly once in the list. We will use the same trick as used in Lecture note 10 to enumerate the elements of $\{0,1\}^*$ We get our bijection by setting $f(n)$ to be the $n$-th string in the list. List all strings of length 1 in lexicographic order, and then all strings of length 2 in lexicographic order, and then strings of length 3 in lexicographic order, and so forth. Since at each step, there are only finitely many strings of a particular length $\ell$, any string of finite length appears in the list. It is also clear that each string appears exactly once in this list.

(e) Countably infinite. Let $\mathscr{A} = \{a_1, a_2, \dots\}$ denote the alphabet. (We are making use of the fact that the alphabet is countably infinite when we assume there is such an enumeration.) We will provide two solutions:

*Alternative 1:* We will enumerate all the strings similar to that in part (b), although the enumeration requires a little more finesse. Notice that if we tried to list all strings of length 1, we would be stuck forever, since the alphabet is infinite! On the other hand, if we try to restrict our alphabet and only print out strings containing the first character $a \in \mathscr{A}$, we would also have a similar problem: the list

$$a, aa, aaa, \dots$$

also does not end.

The idea is to restrict *both* the length of the string and the characters we are allowed to use:

1. List all strings containing only $a_1$ which are of length at most 1.
2. List all strings containing only characters in $\{a_1, a_2\}$ which are of length at most 2 and have not been listed before.
3. List all strings containing only characters in $\{a_1, a_2, a_3\}$ which are of length at most 3 and have not been listed before.
4. Proceed onwards.

At each step, we have restricted ourselves to a finite alphabet with a finite length, so each step is guaranteed to terminate. To show that the enumeration is complete, consider any string $s$ of length $\ell$; since the length is finite, it can contain at most $\ell$ distinct $a_i$ from the alphabet. Let $k$ denote the largest index of any $a_i$ which appears in $s$. Then, $s$ will be listed in step $\max(k, \ell)$, so it appears in the enumeration. Further, since we are listing only those strings that have not appeared before, each string appears exactly once in the listing.

*Alternative 2:* We will encode the strings into ternary strings. Recall that we used a similar trick in Lecture note 10 to show that the set of all polynomials with natural coefficients is countable. Suppose, for example, we have a string: $S = a_5a_2a_7a_4a_6$. Corresponding to each of the characters in this string, we can write its index as a binary string: $(101, 10, 111, 100, 110)$. Now, we can construct a ternary string where "2" is inserted as a separator between each binary string. Thus we map the string $S$ to a ternary string: 101210211121002110. It is clear that this mapping is injective, since the original string $S$ can be uniquely recovered from this ternary string. Thus we have an injective map to $\{0,1,2\}^*$. From Lecture note 10, we know that the set $\{0,1,2\}^*$ is countable, and hence the set of all strings with finite length over $\mathscr{A}$ is countable.

(f) Uncountably infinite. We can use a diagonalization argument. First, for a string $s$, define $s[i]$ as the $i$-th character in the string (where the first character is position 0), where $i \in \mathbb{N}$ because the strings are infinite. Now suppose for contradiction that we have an enumeration of strings $s_i$ for all $i \in \mathbb{N}$: then define the string $s'$ as $s'[i] = (\text{the next character in the alphabet after } s_i[i])$, where the character after $z$ loops around back to $a$. Then $s'$ differs at position $i$ from $s_i$ for all $i \in \mathbb{N}$, so it is not accounted for in the enumeration, which is a contradiction. Thus, the set is uncountable.

*Alternative 1:* The set of all infinite strings containing only $a$s and $b$s is a subset of the set we're counting. We can show a bijection from this subset to the real interval $\mathbb{R}[0,1]$, which proves the uncountability of the subset and therefore entire set as well: given a string in $\{a,b\}^*$, replace the $a$s with 0s and $b$s with 1s and prepend $'0.'$ to the string, which produces a unique binary number in $\mathbb{R}[0,1]$ corresponding to the string.

## 3 Counting Cartesian Products

Note 11

For two sets $A$ and $B$, define the cartesian product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets $A$ and $B$, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \dots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.

(c) Consider a countably infinite number of finite sets: $B_1, B_2, \dots$ for which each set has at least 2 elements. Prove that $B_1 \times B_2 \times \cdots$ is uncountable.

**Solution:**

(a) As shown in lecture, $\mathbb{N} \times \mathbb{N}$ is countable by creating a zigzag map that enumerates through the pairs: $(0,0),(1,0),(0,1),(2,0),(1,1),\dots$. Since $A$ and $B$ are both countable, there exists a bijection between each set and a subset of $\mathbb{N}$. Thus we know that $A \times B$ is countable because there is a bijection between a subset of $\mathbb{N} \times \mathbb{N}$ and $A \times B : f(i,j) = (A_i, B_j)$. We can enumerate the pairs $(a,b)$ similarly.

(b) Proceed by induction.
Base Case: $n = 2$. We showed in part (a) that $A_1 \times A_2$ is countable since both $A_1$ and $A_2$ are countable.
Induction Hypothesis: Assume that for some $n \in \mathbb{N}$, $A_1 \times A_2 \times \cdots \times A_n$ is countable.
Induction Step: Consider $A_1 \times \cdots \times A_n \times A_{n+1}$. We know from our hypothesis that $A_1 \times \cdots \times A_n$ is countable, call it $C = A_1 \times \cdots \times A_n$. We proved in part (a) that since $C$ is countable and $A_{n+1}$ are countable, $C \times A_{n+1}$ is countable, which proves our claim.

(c) Let us assume that each $B_i$ has size 2. If any of the sizes are greater than 2, that would only make the cartesian product larger. Notice that this is equivalent to the set of infinite length binary strings, which was proven to be uncountable in the notes.

Alternatively, we could provide a diagonalization argument: Assuming for the sake of contradiction that $B_1 \times B_2 \times \cdots$ is countable and its elements can be enumerated in a list:

$$\begin{gathered}(b_{1,1}, b_{2,1}, b_{3,1}, b_{4,1}, \dots)\\(b_{1,2}, b_{2,2}, b_{3,2}, b_{4,2}, \dots)\\(b_{1,3}, b_{2,3}, b_{3,3}, b_{4,3}, \dots)\\(b_{1,4}, b_{2,4}, b_{3,4}, b_{4,4}, \dots)\\\vdots\end{gathered}$$

where $b_{i,j}$ represents the item from set $B_i$ that is included in the $j$th element of the Cartesian Product. Now consider the element $(\overline{b_{1,1}}, \overline{b_{2,2}}, \overline{b_{3,3}}, \overline{b_{4,4}}, \dots)$, where $\overline{b_{i,j}}$ represents any item from set $B_i$ that differs from $b_{i,j}$ (i.e. any other element in the set). This is a valid element that should exist in the Cartesian Product $B_1, B_2, \dots$, yet it is not in the enumerated list. This is a contradiction, so $B_1 \times B_2 \times \cdots$ must be uncountable.

# 4 Counting Functions

Note 11

Are the following sets countable or uncountable? Prove your claims.

(a) The set of all functions $f$ from $\mathbb{N}$ to $\mathbb{N}$ such that $f$ is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$.

(b) The set of all functions $f$ from $\mathbb{N}$ to $\mathbb{N}$ such that $f$ is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.

**Solution:**

(a) Uncountable: Let us assume the contrary and proceed with a diagonalization argument. If there are countably many such function we can enumerate them as

| | 0 | 1 | 2 | 3 | ... |
|---|---|---|---|---|---|
| $f_0$ | $f_0(0)$ | $f_0(1)$ | $f_0(2)$ | $f_0(3)$ | ... |
| $f_1$ | $f_1(0)$ | $f_1(1)$ | $f_1(2)$ | $f_1(3)$ | ... |
| $f_2$ | $f_2(0)$ | $f_2(1)$ | $f_2(2)$ | $f_2(3)$ | ... |
| $f_3$ | $f_3(0)$ | $f_3(1)$ | $f_3(2)$ | $f_3(3)$ | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

Now go along the diagonal and define $f$ such that $f(x) > f_x(x)$ and $f(y) > f(x)$ if $y > x$, which is possible because at step $k$ we only need to find a number $\in \mathbb{N}$ greater than all the $f_j(j)$ for $j \in \{0, \ldots, k\}$; for example, we could define such a function using

$$f(x) = \begin{cases} f_0(0) + 1 & x = 0 \\ \max(f_x(x), f(x-1)) + 1 & x > 0 \end{cases}$$

This function differs from each $f_i$ and therefore cannot be on the list, hence the list does not exhaust all non-decreasing functions. As a result, there must be uncountably many such functions.

*Alternative Solution:* Look at the subset $\mathscr{S}$ of strictly increasing functions. Any such $f$ is uniquely identified by its image which is an infinite subset of $\mathbb{N}$. But the set of infinite subsets of $\mathbb{N}$ is uncountable. This is because the set of all subsets of $\mathbb{N}$ is uncountable, and the set of all finite subsets of $\mathbb{N}$ is countable. So $\mathscr{S}$ is uncountable and hence the set of all non-decreasing functions must be too.

*Alternative Solution 2:* We can inject the set of infinitely long binary strings into the set of non-decreasing functions as follows. For any infinitely long binary string $b$, let $f(n)$ be equal to the number of 1's appearing in the first $n$-digits of $b$. It is clear that the function $f$ so defined is non-decreasing. Also, since the function $f$ is uniquely defined by the infinitely long binary string, the mapping from binary strings to non-decreasing functions is injective. Since the set of infinite binary strings is uncountable, and we produced an injection from that set to the set of non-decreasing functions, that set must be uncountable as well.

(b) Countable: Let $D_n$ be the subset of non-increasing functions for which $f(0) = n$. Any such function must stop decreasing at some point (because $\mathbb{N}$ has a smallest number), so there can only be finitely many (at most $n$) points $X_f = \{x_1, \ldots, x_k\}$ at which $f$ decreases. Let $y_i$ be the amount by which $f$ decreases at $x_i$, then $f$ is fully described by $\{(x_1, y_1), \ldots, (x_k, y_k),$

$(-1,0),\ldots,(-1,0)\} \in \mathbb{N}^n = \mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}$ ($n$ times), where we padded the $k$ values associated with $f$ with $n-k$ $(-1,0)$s. In Lecture note 11, we have seen that $\mathbb{N}\times\mathbb{N}$ is countable by the spiral method. Using it repeatedly, we get $\mathbb{N}^{(2^l)}$ is countable for all $l \in \mathbb{N}$. This gives us that $\mathbb{N}^n$ is countably for any finite $n$ (because $\mathbb{N}^n \subset \mathbb{N}^{(2^l)}$ where $l$ is such that $2^l \geq n$). Hence $D_n$ is countable. Since each set $D_n$ is countable we can enumerate it. Map an element of $D_n$ to $(n, j)$ where $j$ is the label of that element produced by the enumeration of $D_n$. This produces an injective map from $\cup_{n\in\mathbb{N}} D_n$ to $\mathbb{N}\times\mathbb{N}$ and we know that $\mathbb{N}\times\mathbb{N}$ is countable from Lecture note 11 (via spiral method). Now the set of all non-increasing functions is $\cup_{i\in\mathbb{N}} D_n$, and thus countable.