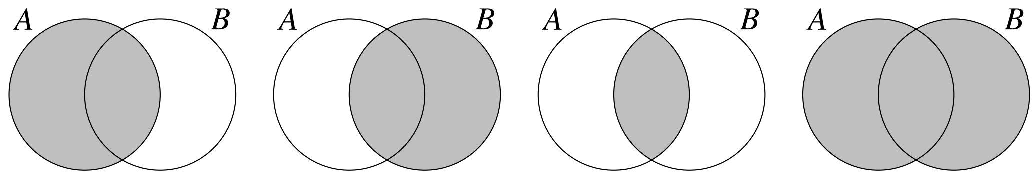
Counting Intro II

Inclusion-exclusion: With two sets,



$$|A| + |B| - |A \cap B| = |A \cup B|$$

With more sets,

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = &|A_1| + |A_2| + \cdots + |A_n| \\ &- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_i \cap A_j| - \cdots - |A_{n-1} \cap A_n| \\ &+ |A_1 \cap A_2 \cap A_3| + \cdots + |A_i \cap A_j \cap A_k| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \\ &\cdots\end{aligned}$$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=k}} \left|\bigcap_{i \in S} A_i\right|$$

That is, for each size k, iterate through all ways of picking k sets from $\{A_1, \ldots, A_n\}$, and alternate between adding and subtracting the sizes of their intersection.

Combinatorial proofs: A technique for proving combinatorial identities. There should be very little math involved (usually none): use two different ways of counting the same scenario. One way should correspond to the left-hand side of the equality, and the other way should correspond to the right-hand side of the equality. The fact that we're counting the same scenario means that the two sides are equal.

1 Inclusion and Exclusion

Note 10 What is the total number of positive integers strictly less than 100 that are also coprime to 100?

Solution: It is sufficient to count the opposite: what is the total number of positive integers strictly less than 100 and *not* coprime to 100?

If a number is not coprime to 100, this means that the number is either a multiple of 2 or a multiple of 5. In this case, we have:

- 49 multiples of 2
- 19 multiples of 5
- 9 multiples of both 2 and 5

By inclusion-exclusion, the total number of positive integers not coprime to 100 is $49+19-9=59$, and there are 99 positive integers strictly less than 100.

As such, in total there are $99-59=40$ different positive integers strictly less than 100 that are coprime to 100.

2 CS70: The Musical

Note 10

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(b) Edward would now like to select a crew out of n people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}.$$

Solution:

(a) Say that we would like to select 2 directors.

LHS: This is the number of ways to choose 2 directors out of the $2n$ candidates.

RHS: Split the $2n$ directors into two groups of n; one group consisting of experienced directors, or inexperienced directors (you can split arbitrarily). Then, we consider three cases: either we choose:

(a) Both directors from the group of experienced directors,

(b) Both directors from the group of inexperienced directors, or

(c) One experienced director and one inexperienced director.

The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2, respectively. Since these cases are mutually exclusive and cover all possibilities, it must also count the total number of ways to choose 2 directors out of the $2n$ candidates. This completes the proof.

(b) Say that we would like to select k crew members.

LHS: This is simply the number of ways to choose k crew members out of n candidates.

RHS: We select the k crew members in a different way. First, Edward looks at the first candidate he sees and decides whether or not he would like to choose the candidate. If he selects the first candidate, then Edward needs to choose $k-1$ more crew members from the remaining $n-1$ candidates. Otherwise, he needs to select all k crew members from the remaining $n-1$ candidates.

We are not double counting here - since in the first case, Edward takes the first candidate he encounters, and in the other case, we do not.

(c) In this part, Edward selects a subset of the n actors to be in his musical. Additionally, assume that he must select one individual as the lead for his musical.

LHS: Edward casts k actors in his musical, and then selects one lead among them (note that $k = \binom{k}{1}$). The summation is over all possible sizes for the cast - thus, the expression accounts for all subsets of the n actors.

RHS: From the n people, Edward selects one lead for his musical. Then, for the remaining $n-1$ actors, he decides whether or not he would like to include them in the cast. 2^{n-1} represents the amount of (possibly empty) subsets of the remaining actors. *(Note that for each actor, Edward has 2 choices: to include them, or to exclude them.)*

(d) In this part, Edward selects a subset of the n actors to be in the musical; additionally he must select j lead actors (instead of only 1 in the previous part).

LHS: Edward casts $k \geq j$ actors in his musical, then selects the j leads among them. Again, the summation is over all possible sizes for the cast (note that any cast that has $< j$ members is invalid, since Edward would be unable to select j lead actors) - thus, the expression accounts for all valid subsets of the n actors.

RHS: From the n people, Edward selects j leads for his musical. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. 2^{n-j} represents the amount of ways that Edward can do this.

3 Countability Intro

Countability: Formal notion of different kinds of infinities.

- *Countable*: able to enumerate in a list (possibly finite, possibly infinite)
- *Countably infinite*: able to enumerate in an infinite list; that is, there is a bijection with $\mathbb{N}$.

  To show that there is a bijection, the *Cantor–Bernstein theorem* says that it is sufficient to find two injections, $f : S \to \mathbb{N}$ and $g : \mathbb{N} \to S$. Intuitively, this is because an injection $f : S \to \mathbb{N}$ means $|S| \leq |\mathbb{N}|$, and an injection $g : \mathbb{N} \to S$ means $|\mathbb{N}| \leq |S|$; together, we have $|\mathbb{N}| = |S|$.
- *Uncountably infinite*: unable to be listed out

  Use *Cantor diagonalization* to prove uncountability through contradiction; the classic example is the set of reals in $[0,1]$:

$\mathbb{N}$	$[0,1]$
0	0 . ⑦ 3 2 0 5 0 ⋯
1	0 . 4 ① 4 2 1 3 ⋯
2	0 . 6 1 ⑧ 0 3 3 ⋯
3	0 . 1 8 2 ⑧ 1 8 ⋯
4	0 . 1 4 1 5 ⑨ 2 ⋯
5	0 . 5 7 7 2 1 ⑤ ⋯
⋮	⋮
?	0 . 8 2 9 9 1 6 ⋯

  If we change the digits along the diagonal, the new decimal created is different from every single element in the list in at least one place, so it's not in the list—this is a contradiction.

Sometimes it can be easier to prove countability/uncountability through bijections with other countable/uncountable sets respectively. Common countable sets include $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N} \times \mathbb{N}$, finite length bitstrings, etc. Common uncountable sets include $[0,1]$, $\mathbb{R}$, infinite length bitstrings, etc.

(a) Your friend is confused about how Cantor diagonalization doesn't apply to the set of natural numbers. They argue that natural numbers can be thought of as an infinite length string of digits, by padding each number with an infinite number of zeroes to the left. If we then assume by contradiction that we can list out the set of natural numbers with the padded 0's, we can change the digits along a diagonal, to create a new natural number not in the list.

$$
\begin{array}{c|ccccc}
0 & \cdots\ 0 & 0 & 0 & 0 & \textcircled{1} \\
1 & \cdots\ 0 & 1 & 2 & \textcircled{3} & 4 \\
2 & \cdots\ 5 & 2 & \textcircled{8} & 2 & 3 \\
3 & \cdots\ 9 & \textcircled{4} & 3 & 2 & 1 \\
\vdots & & & \vdots & & \\
\hline
? & \cdots\ 1 & 5 & 9 & 4 & 2
\end{array}
$$

What is wrong with this argument?

(b) Classify the following sets as either countable or uncountable, with brief justification.

(i) The set of integers x that solve the equation $3x \equiv 2 \pmod{10}$.

(ii) The set of real solutions for the equation $x + y = 1$.

(iii) The set of all irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ (i.e. real numbers that are not rational).

Solution:

(a) The issue here is that the newly created number is not necessarily a natural number. This number has an infinite number of digits (we'll always be changing the padded zeroes into some nonzero digits), while natural numbers must have a finite number of digits.

This means that when you perform Cantor diagonalization to show a set S is uncountable, it is imperative that you ensure that the newly created element is always still an element of S; otherwise, we cannot say anything about the result of the diagonalization.

(b) (i) Countable. The solution to the equation is of the form

$$
\begin{aligned}
3x &\equiv 2 \pmod{10} \\
x &\equiv 3^{-1} \cdot 2 \pmod{10} \\
&\equiv 7 \cdot 2 \equiv 14 \equiv 4 \pmod{10}
\end{aligned}
$$

The integers that are equivalent to $4 \pmod{10}$ are in the set $\{\ldots, -16, -6, 4, 14, \ldots\}$, which is countable, since it is a subset of $\mathbb{Z}$.

We also technically do not need to explicitly solve for this set; we already know that the solution will be some subset of the integers. We've shown in the notes that $\mathbb{Z}$ is countable, so any subset of $\mathbb{Z}$ must also be countable.

(ii) Uncountable. For any fixed $x \in \mathbb{R}$, there is exactly one solution to $x + y = 1$, namely $y = 1 - x$. This means that we have a bijection to $\mathbb{R}$ by simply taking the x-coordinate of the solution. Since we know that $\mathbb{R}$ is uncountable, this set of solutions is also uncountable.

(iii) Uncountable. Suppose for contradiction that $\mathbb{R} \setminus \mathbb{Q}$ is countable. We know that $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable.

Notice that we can write $(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$. Since $\mathbb{R} \setminus \mathbb{Q}$ is countable, and $\mathbb{Q}$ is also countable, their union must also be countable (we can list the union by alternating between the lists for $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{Q}$). However, this is a contradiction—the union is $\mathbb{R}$, and we know that it is uncountable.

This means that $\mathbb{R} \setminus \mathbb{Q}$ must be uncountable.