CS 70 Fall 2024	Discrete Mathematics and Probability Theory Hug, Rao	DIS 8A

Discrete Probability Intro

Note 13

Probability Space: A probability space is a tuple $(\Omega, \mathbb{P})$, where Ω is the *sample space* and $\mathbb{P}$ is the *probability function* on the sample space.

Specifically, Ω is the set of all outcomes ω, and $\mathbb{P}$ is a function $\mathbb{P}\colon \Omega \to [0,1]$, assigning a probability to each outcome, satisfying the following conditions:

$$0 \le \mathbb{P}[\omega] \le 1 \quad \text{and} \quad \sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1.$$

Event: an event A is a subset of Ω, i.e. a collection of some outcomes in the sample space. We define

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega].$$

Uniform Probability Space: all outcomes are assigned the same probability, i.e. $\mathbb{P}[\omega] = \frac{1}{|\Omega|}$; this is just counting!

With an event A in a uniform probability space, $\mathbb{P}[A] = \frac{|A|}{|\Omega|}$, which is again more counting!

1 Venn Diagram

Note 13

Out of 1,000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.

(a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.

(b) What is the probability that the student belongs to a club?

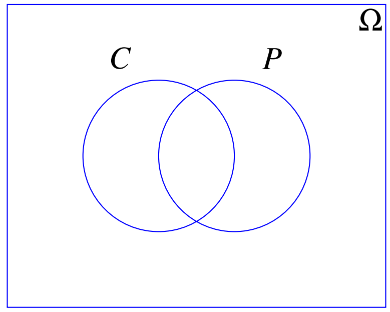
(c) What is the probability that the student works part time?

(d) What is the probability that the student belongs to a club AND works part time?

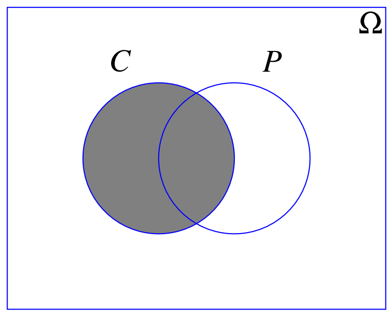
(e) What is the probability that the student belongs to a club OR works part time?

Solution:

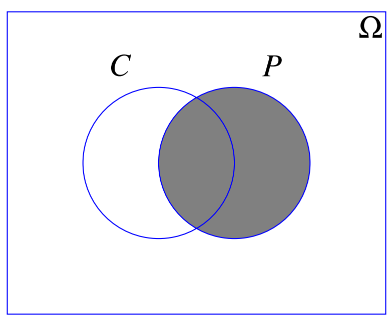
(a) The sample space will be illustrated by a Venn diagram.



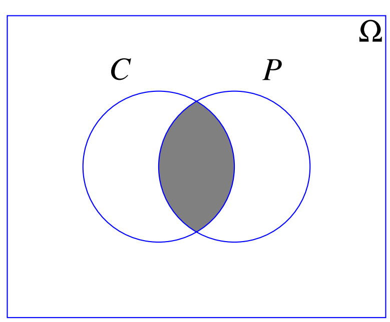
(b) $\mathbb{P}[C] = \frac{|C|}{|\Omega|} = \frac{400}{1000} = \frac{2}{5}$.



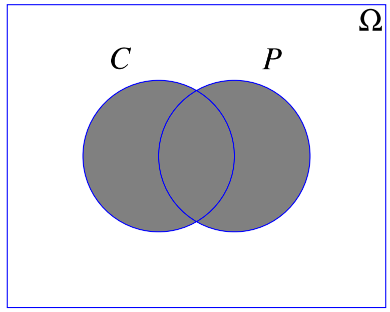
(c) $\mathbb{P}[P] = \frac{|P|}{|\Omega|} = \frac{500}{1000} = \frac{1}{2}$.



(d) $\mathbb{P}[P \cap C] = \frac{|P \cap C|}{|\Omega|} = \frac{50}{1000} = \frac{1}{20}$.



(e) $\mathbb{P}[P \cup C] = \mathbb{P}[P] + \mathbb{P}[C] - \mathbb{P}[P \cap C] = \frac{1}{2} + \frac{2}{5} - \frac{1}{20} = \frac{17}{20}$.



2 Flippin' Coins

Note 13

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

(g) What is the probability of the event that our outcome has at least one head?

Solution:

(a) $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

(b) An event must be a subset of Ω, meaning that it must consist of possible outcomes.

- No

- Yes
- Yes
- Yes
- Yes

(c) $\{(T,H,H),(T,H,T),(T,T,H)\}$

(d) $\{(T,H,H),(H,H,T),(H,T,H),(T,T,T)\}$

(e) Since $|\Omega| = 2^3 = 8$ and every outcome has equal probability, $\mathbb{P}[(H,H,T)] = 1/8$.

(f) The event of interest is $E = \{(H,H,T),(H,T,H),(T,H,H)\}$, which has size 3. Whence $\mathbb{P}[E] = 3/8$.

(g) If we do not see at least one head, then we must see at exactly three tails. The event $\overline{E} = \{(T,T,T)\}$ of seeing exactly three tails is thus the complement of the event E that we see at least one head. $\overline{E}$ occurs with probability $(1/2)^3 = 1/8$, so its complement E must occur with probability $1 - 1/8 = 7/8$.

3 Sampling

Note 13

Suppose you have balls numbered $1,\dots,n$, where n is a positive integer ≥ 2, inside a coffee mug. You pick a ball uniformly at random, look at the number on the ball, replace the ball back into the coffee mug, and pick another ball uniformly at random.

(a) What is the probability that the first ball is 1 and the second ball is 2?

(b) What is the probability that the second ball's number is strictly less than the first ball's number?

(c) What is the probability that the second ball's number is exactly one greater than the first ball's number?

(d) Now, assume that after you looked at the first ball, you did *not* replace the ball in the coffee mug (instead, you threw the ball away), and then you drew a second ball as before. Now, what are the answers to the previous parts?

Solution:

(a) Out of n^2 pairs of balls that you could have chosen, only one pair $(1,2)$ corresponds to the event we are interested in, so the probability is $1/n^2$.

(b) Again, there are n^2 total outcomes. Now, we want to count the number of outcomes where the second ball's number is strictly less than the first ball's number. Similarly to the last part, we can view any outcome as an ordered pair (n_1,n_2), where n_1 is the number on the first ball, and

n_2 is the number on the second ball. There are $\binom{n}{2}$ outcomes where $n_1 > n_2$; select two distinct numbers from $[1,n]$, and assign the higher number to n_1. Thus, the probability is $\frac{\binom{n}{2}}{n^2} = \frac{n-1}{2n}$.

Alternate Solution: The probability that the two balls have the same number is $n/n^2 = 1/n$, so the probability that the balls have different numbers is $1 - 1/n = (n-1)/n$. By symmetry, it is equally likely for the first ball to have a greater number and for the second ball to have a greater number, so we take the probability $(n-1)/n$ and divide it by two to obtain $(n-1)/(2n)$.

(c) Again, there are n^2 pairs of balls that we could have drawn, but there are $n-1$ pairs of balls which correspond to the event we are interested in: $\{(1,2),(2,3),\ldots,(n-1,n)\}$. So, the probability is $(n-1)/n^2$.

(d) There are a total of $n(n-1)$ pairs of balls that we could have drawn, and only the pair $(1,2)$ corresponds to the event that we are interested in, so the probability is $1/(n(n-1))$.

The probability that the two balls are the same is now 0, but the symmetry described earlier still applies, so the probability that the second ball has a smaller number is $1/2$.

There are a total of $n(n-1)$ pairs of balls that we could have drawn, and we are interested in the $n-1$ pairs $(1,2),(2,3),\ldots,(n-1,n)$ as before. Thus, the probability that the second ball is one greater than the first ball is $1/n$.