Combinations of Events Intro

Note 14

Product rule: We can find the probability of an intersection of events by enforcing an "ordering" of these events. Here, each successive conditional probability in the product finds the probability of the next event, *conditioned* on all prior events occurring:

$$\mathbb{P}[A_1 \cap A_2 \cap \cdots \cap A_n] = \mathbb{P}[A_1]\mathbb{P}[A_2 \mid A_1]\mathbb{P}[A_3 \mid A_1 \cap A_2] \cdots \mathbb{P}[A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}].$$

Note that this is just a generalization of the definition of conditional probability: $\mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_1]\mathbb{P}[A_2 \mid A_1]$

Union Bound: Derived from the principle of inclusion-exclusion, the probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs is at most the sum of the probabilities of the individual events:

$$\mathbb{P}[A_1 \cup A_2 \cup \cdots \cup A_n] \leq \mathbb{P}[A_1] + \mathbb{P}[A_2] + \cdots + \mathbb{P}[A_n]$$
$$\mathbb{P}[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} \mathbb{P}[A_i]$$

with equality when the A_i's are disjoint.

1 Symmetry

Note 11
Note 13

In this problem, we will walk you through the idea of *symmetry* and its formal justification. Consider an experiment where you have a bag with m red marbles and $n-m$ blue marbles. You draw marbles from the bag, one at a time without replacement until the bag is empty.

(a) Define the sample space Ω. (No need to write out every element, a brief description is fine). Is this a uniform probability space?

(b) What is the probability that the first marble you draw is red?

(c) Suppose you've drawn all but the final marble, setting each marble aside as you draw it *without looking at it*. We want to find the probability that the final marble left in the bag will be red.

Let A be the event containing outcomes where the first marble is red, and let B be the event containing outcomes where the final marble is red. Provide a bijective function $f : A \to B$ mapping outcomes in A to outcomes in B, and explain why it is a bijection.

(d) Use the previous parts to find the probability that the final marble will be red.

(e) You repeat the experiment. Find the probability that the last two marbles you draw will be red.

(f) You repeat the experiment again, but this time you see that the first marble you draw is red. Find the probability that the second-to-last marble you draw will also be red.

Solution:

(a) The sample space is the set of all length n sequences with m reds and $n-m$ blues. This is a uniform probability space; there are a total of $\binom{n}{m}$ outcomes in the sample space, and each outcome has probability $\frac{1}{\binom{n}{m}}$.

(b) Of the n marbles, m are red, giving a probability of $\frac{m}{n}$.

(c) The inputs to f will be sequences of length n with m red draws and $n-m$ blue draws, and the output will be the same sequence except with the first and last draws swapped. This uniquely transforms each sequence of draws with a red marble first into a sequence with a red marble last, and vice versa (f is its own inverse).

(d) Since we have a bijection between the events A and B, they have the same number of outcomes. Additionally, we have a uniform probability space, so $\mathbb{P}[A] = \frac{|A|}{|\Omega|}$ and $\mathbb{P}[B] = \frac{|B|}{|\Omega|}$. But our bijection showed that $|A| = |B|$, so $\mathbb{P}[A] = \mathbb{P}[B]$, which means the probability of drawing a red marble last is the same as the probability of drawing a red marble first, which is $\frac{m}{n}$.

Note: We don't require a uniform probability space in order to apply the idea of symmetry. The mapping f only needs to map outcomes in A to outcomes in B with the same probability. Mathematically, we require that for every $\omega \in A$, we have $\mathbb{P}[\omega] = \mathbb{P}[f(\omega)]$. Then we'd have

$$\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega] = \sum_{\omega \in A} \mathbb{P}[f(\omega)] = \sum_{\omega \in B} \mathbb{P}[\omega] = \mathbb{P}[B]$$

as desired.

(e) By the same logic as before, the probability that the last two marbles are red is the same as the probability that the first two marbles are red, which is $\frac{m}{n} \times \frac{m-1}{n-1} = \frac{m(m-1)}{n(n-1)}$. The explicit bijection swaps the first two marbles with the last two marbles.

(f) After seeing the first marble is red, there are $m-1$ red marbles left and $n-1$ total marbles. By symmetry, the probability that the second-to-last marble will be red is the same as the probability that the second marble will be red, which is $\frac{m-1}{n-1}$.

2 Balls and Bins

Note 14 Suppose you throw n balls into n labeled bins one at a time.

(a) What is the probability that the first bin is empty?

(b) What is the probability that the first k bins are empty?

(c) Let A be the event that at least k bins are empty. Let m be the number of subsets of k bins out of the total n bins. If we assume A_i is the event that the ith set of k bins is empty. Then we can write A as the union of A_i's:

$$A = \bigcup_{i=1}^{m} A_i.$$

Compute m in terms of n and k, and use the union bound to give an upper bound on the probability $\mathbb{P}[A]$.

(d) What is the probability that the second bin is empty given that the first one is empty?

(e) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

(f) Are the events that "the first bin is empty" and "the second bin is empty" independent?

Solution: Since the balls are thrown one at a time, there is an ordering, and so we are sampling with replacement where order matters rather than where it doesn't (which would correspond to each configuration in the stars and bars setup being equally likely).

(a) Note that this is a uniform sample space, with outcomes representing all possible ways to throw each ball individually into the bins. Here, $|\Omega| = n^n$, as each of the n balls has n possible bins to fall into, and out of these possibilities, $(n-1)^n$ of them leave the first bin empty—each ball would then have $n-1$ possible bins to fall into. This gives us an overall probability $\left(\frac{n-1}{n}\right)^n$ that the first bin is empty.

Equivalently, we can note that each throw is independent of all of the other throws. Since the probability that ball i does not land in the first bin is $\frac{n-1}{n}$, the probability that all of the balls do not land in the first bin is $\left(\frac{n-1}{n}\right)^n$.

(b) Similar to the previous part, we have the same uniform sample space of size n^n. Now, there are a total of $(n-k)^n$ possible ways to throw the balls into bins such that the first k bins are empty—each ball has $n-k$ possible bins to fall into.

Alternatively, we can similarly make use of independence. Since the probability that ball i does not land in the first k bins is $\frac{n-k}{n}$, the probability that all of the balls do not land in the first k bins is $\left(\frac{n-k}{n}\right)^n$.

(c) We use the union bound. Then

$$\mathbb{P}[A] = \mathbb{P}\left[\bigcup_{i=1}^{m} A_i\right] \leq \sum_{i=1}^{m} \mathbb{P}[A_i].$$

We know the probability of the first k bins being empty from part (b), and this is true for any set of k bins, so

$$\mathbb{P}[A_i] = \left(\frac{n-k}{n}\right)^n.$$

Then,

$$\mathbb{P}[A] \le m \cdot \left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n.$$

(d) Using Bayes' Rule:

$$\begin{aligned}\mathbb{P}[\text{2nd bin empty} \mid \text{1st bin empty}] &= \frac{\mathbb{P}[\text{2nd bin empty} \cap \text{1st bin empty}]}{\mathbb{P}[\text{1st bin empty}]} \\ &= \frac{(n-2)^n/n^n}{(n-1)^n/n^n} \\ &= \left(\frac{n-2}{n-1}\right)^n\end{aligned}$$

Alternate solution: We know bin 1 is empty, so each ball that we throw can land in one of the remaining $n-1$ bins. We want the probability that bin 2 is empty, which means that each ball cannot land in bin 2 either, leaving $n-2$ bins. Thus for each ball, the probability that bin 2 is empty given that bin 1 is empty is $\frac{n-2}{n-1}$. For n total balls, this probability is $\left(\frac{n-2}{n-1}\right)^n$.

(e) They are dependent. Knowing the latter means the former happens with probability 1.

(f) In part (c) we calculated the probability that the second bin is empty given that the first bin is empty: $\left(\frac{n-2}{n-1}\right)^n$. The probability that the second bin is empty (without any prior information) is $\left(\frac{n-1}{n}\right)^n$. Since these probabilities are not equal, the events are dependent.

3 Mario's Coins

Note 14

Mario owns three identical-looking coins. One coin shows heads with probability $1/4$, another shows heads with probability $1/2$, and the last shows heads with probability $3/4$.

(a) Mario randomly picks a coin and flips it. He then picks one of the other two coins and flips it. Let X_1 and X_2 be the events of the 1st and 2nd flips showing heads, respectively. Are X_1 and X_2 independent? Please prove your answer.

(b) Mario randomly picks a single coin and flips it twice. Let Y_1 and Y_2 be the events of the 1st and 2nd flips showing heads, respectively. Are Y_1 and Y_2 independent? Please prove your answer.

(c) Mario arranges his three coins in a row. He flips the coin on the left, which shows heads. He then flips the coin in the middle, which shows heads. Finally, he flips the coin on the right. What is the probability that it also shows heads?

Solution:

(a) X_1 and X_2 are not independent. Intuitively, the fact that X_1 happened gives some information about the first coin that was chosen; this provides some information about the second coin that was chosen (since the first and second coins can't be the same coin), which directly affects whether X_2 happens or not.

To make this formal, we compute

$$\mathbb{P}[X_1] = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

By symmetry, $\mathbb{P}[X_2] = \mathbb{P}[X_1]$, so

$$\mathbb{P}[X_1]\,\mathbb{P}[X_2] = \frac{1}{4}.$$

But if we consider the probability that both X_1 and X_2 happen, we have

$$\begin{aligned}\mathbb{P}[X_1 \cap X_2] &= \frac{1}{6}\Big[\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \\ &\qquad \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\Big] \\ &= \frac{22}{96} = \frac{11}{48}\end{aligned}$$

which is not equal to $1/4$, violating the definition of independence.

(b) Y_1 and Y_2 are not independent. Intuitively, the fact that Y_1 happens gives some information about the coin that was picked, which directly influences whether Y_2 happens or not.

To make this formal, we compute

$$\mathbb{P}[Y_1] = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{2}$$

By symmetry, $\mathbb{P}[Y_2] = \mathbb{P}[Y_1]$, so

$$\mathbb{P}[Y_1]\,\mathbb{P}[Y_2] = \frac{1}{4}$$

But if we consider the probability that both Y_1 and Y_2 happen, we have

$$\mathbb{P}[Y_1 \cap Y_2] = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{3}{4}\right)^2 = \frac{14}{48} = \frac{7}{24}$$

which is not equal to $1/4$, violating the definition of independence.

(c) Let A be the coin with bias $1/4$, B be the fair coin, and C be the coin with bias $3/4$. There are six orderings, each with probability $1/6$: ABC, ACB, BAC, BCA, CAB, and CBA. Thus

$$\begin{aligned}
&\mathbb{P}[\text{Third coin shows heads} \mid \text{First two coins show heads}] \\
&= \frac{\mathbb{P}[\text{All three coins show heads}]}{\mathbb{P}[\text{First two coins show heads}]} \\
&= \frac{(\frac{1}{4})(\frac{1}{2})(\frac{3}{4})}{11/48} \\
&= \frac{3/32}{11/48} = \frac{9}{22}.
\end{aligned}$$

Note that the denominator was the probability calculated in part a, so we just plugged it in as $\frac{11}{48}$.