Covariance and Total Expectation Intro

Covariance: measure of the relationship between two RVs

$$\operatorname{cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y].$$

The sign of $\operatorname{cov}(X,Y)$ illustrates how X and Y are related; a positive value means that X and Y tend to increase and decrease together, while a negative value means that X increases as Y decreases (and vice versa). A covariance of zero means that the two random variables are uncorrelated—there is no relationship between them.

Properties: for random variables X, Y, Z and constant a,

- $\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{cov}(X,Y)$
- $\operatorname{cov}(X,X) = \operatorname{Var}(X)$
- $\operatorname{cov}(X,Y) = \operatorname{cov}(Y,X)$
- Bilinearity: $\operatorname{cov}(X+Y,Z) = \operatorname{cov}(X,Z) + \operatorname{cov}(Y,Z)$ and $\operatorname{cov}(aX,Y) = a\operatorname{cov}(X,Y)$

Conditional Expectation: When we want to find the expectation of a random variable X conditioned on an event A, we use the following formula:

$$\mathbb{E}[X \mid A] = \sum_x x \cdot \mathbb{P}[(X=x) \mid A].$$

This is an application of the definition of expectation. We still consider all values of X but reweigh them based on their probability of occurring together with A.

Total Expectation: For any random variable X and events $A_1, A_2, \dots, A_n$ that partition the sample space Ω,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i]\,\mathbb{P}[A_i].$$

We can think of this as splitting the sample space into partitions (events) and looking at the expectation of X in each partition, weighted by the probability of that event occurring.

1 Covariance

Note 16

(a) We have a bag of 5 red and 5 blue balls. We take two balls uniformly at random from the bag without replacement. Let X_1 and X_2 be indicator random variables for the events of the first and second ball being red, respectively. What is $\operatorname{cov}(X_1,X_2)$? Recall that $\operatorname{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$.

(b) Now, we have two bags A and B, with 5 red and 5 blue balls each. Draw a ball uniformly at random from A, record its color, and then place it in B. Then draw a ball uniformly at random from B and record its color. Let X_1 and X_2 be indicator random variables for the events of the first and second draws being red, respectively. What is $\text{cov}(X_1, X_2)$?

Solution:

(a) We can use the formula $\text{cov}(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2]$.

$$\mathbb{E}[X_1] = \frac{5}{10} \times 1 + \frac{5}{10} \times 0 = \frac{1}{2},$$
$$\mathbb{E}[X_2] = \frac{5}{10} \times 1 + \frac{5}{10} \times 0 = \frac{1}{2},$$
$$\mathbb{E}[X_1 X_2] = \frac{5}{10} \cdot \frac{4}{9} \times 1 + \left(1 - \frac{5}{10} \cdot \frac{4}{9}\right) \times 0 = \frac{2}{9}.$$

Therefore,

$$\text{cov}(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2] = \frac{2}{9} - \frac{1}{2} \times \frac{1}{2} = -\frac{1}{36}.$$

(b) Again, we use the formula $\text{cov}(X_1, X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]$.

$$\mathbb{E}[X_1] = \frac{5}{10} \times 1 + \frac{5}{10} \times 0 = \frac{1}{2}$$
$$\mathbb{E}[X_2] = \left(\frac{5}{10} \times \frac{6}{11} + \frac{5}{10} \times \frac{5}{11}\right) \times 1 + \left(\frac{5}{10} \times \frac{5}{11} + \frac{5}{10} \times \frac{6}{11}\right) \times 0 = \frac{1}{2}$$
$$\mathbb{E}[X_1 X_2] = \frac{5}{10} \times \frac{6}{11} \times 1 = \frac{30}{110}.$$

Therefore,

$$\mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] = \frac{30}{110} - \frac{1}{4} = \frac{1}{44}.$$

Note that in part (a), if one event happened, the other would be less likely to happen, and thus the covariance was negative. Similarly, in part (b), if one event happened, the other would be more likely to happen, and thus the covariance was positive.

2 Number Game

Note 20

Sinho and Vrettos are playing a game where they each choose an integer uniformly at random from $[0, 100]$, then whoever has the larger number wins (in the event of a tie, they replay). However, Vrettos doesn't like losing, so he's rigged his random number generator such that it instead picks randomly from the integers between Sinho's number and 100. Let S be Sinho's number and V be Vrettos' number.

(a) What is $\mathbb{E}[S]$?

(b) What is $\mathbb{E}[V \mid S = s]$, where s is any constant such that $0 \leq s \leq 100$?

(c) What is $\mathbb{E}[V]$?

Solution:

(a) S is a (discrete) uniform random variable between 0 and 100, so its expectation is $\frac{0+100}{2} = 50$.

(b) If $S = s$, we know that V will be uniformly distributed between s and 100. Similar to the previous part, this gives us that $\mathbb{E}[V \mid S = s] = \frac{s+100}{2}$.

(c) With the law of total expectation, we have that

$$\begin{aligned}\mathbb{E}[V] &= \sum_{s=0}^{100} \mathbb{E}[V \mid S = s] \cdot \mathbb{P}[S = s] \\ &= \sum_{s=0}^{100} \frac{s+100}{2} \cdot \frac{1}{101} \\ &= \frac{1}{202} \left(\sum_{s=0}^{100} s + \sum_{s=0}^{100} 100 \right)\end{aligned}$$

The first summation comes out to $\frac{100(100+1)}{2} = 50 \cdot 101$; the second summation is just adding 100 to itself 101 times, so it comes out to $100 \cdot 101$. Plugging these values in, we get $\mathbb{E}[V] = 75$.

3 Dice Games

Note 20

Suppose you roll a fair six-sided die. You read off the number showing on the die, then flip that many fair coins.

(a) If the result of your die roll is i, what is the expected number of heads you see?

(b) What is the expected number of heads you see?

Solution:

(a) The number of heads you get is binomially distributed with parameters i and $\frac{1}{2}$. Thus, the expected number of heads you see is $\frac{i}{2}$.

(b) Let D be the outcome of the die roll and H be the number of heads you get. We have that

$$\begin{aligned}\mathbb{E}[H] &= \sum_{i=1}^{6} \mathbb{E}[H|D = i] \cdot \mathbb{P}[D = i] \\ &= \sum_{i=1}^{6} \frac{i}{2} \cdot \frac{1}{6} \\ &= \frac{1}{12} \sum_{i=1}^{6} i\end{aligned}$$

We know that $\sum_{i=1}^{n} i$ comes out to $\frac{n(n+1)}{2}$, so $\mathbb{E}[H] = \frac{1}{12} \cdot \frac{6 \cdot 7}{2} = \frac{7}{4}$.

4 Number of Ones

Note 20

In this problem, we will revisit dice-rolling, except with conditional expectation. (*Hint*: for both of these subparts, the law of total expectation may be helpful.)

(a) If we roll a die until we see a 6, how many ones should we expect to see?

(b) If we roll a die until we see a number greater than 3, how many ones should we expect to see?

Solution:

(a) Let Y be the number of ones we see. Let X be the number of rolls we take until we get a 6.

Let us first compute $\mathbb{E}[Y \mid X = k]$. We know that in each of our $k-1$ rolls before the kth, we necessarily roll a number in $\{1, 2, 3, 4, 5\}$. Thus, we have a $1/5$ chance of getting a one in each of these $k-1$ previous rolls, giving

$$\mathbb{E}[Y \mid X = k] = \frac{1}{5}(k-1).$$

If this is confusing, we can write Y as a sum of indicator variables, $Y = Y_1 + Y_2 + \cdots + Y_k$, where Y_i is 1 if we see a one on the ith roll. This means that by linearity of expectation,

$$\mathbb{E}[Y \mid X = k] = \mathbb{E}[Y_1 \mid X = k] + \mathbb{E}[Y_2 \mid X = k] + \cdots + \mathbb{E}[Y_k \mid X = k].$$

We know that on the kth roll, we must roll a 6, so $\mathbb{E}[Y_k] = 0$. Further, by symmetry, each term in this summation has the same value; this means that we have

$$\begin{aligned}
\mathbb{E}[Y_1 \mid X = k] + \mathbb{E}[Y_2 \mid X = k] + \cdots + \mathbb{E}[Y_{k-1} \mid X = k] &= (k-1)\,\mathbb{E}[Y_1 \mid X = k] \\
&= (k-1)\,\mathbb{P}[Y_1 = 1 \mid X = k] \\
&= (k-1)\frac{1}{5}.
\end{aligned}$$

Using the law of total expectation, we now have

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{k=1}^{\infty} \mathbb{E}[Y \mid X = k]\,\mathbb{P}[X = k] && \text{(total expectation)} \\
&= \sum_{k=1}^{\infty} \frac{1}{5}(k-1)\,\mathbb{P}[X = k]
\end{aligned}$$

Here, we can see that this is an application of LOTUS for $f(X) = \frac{1}{5}(X-1)$, so we can simplify this to

$$\begin{aligned}
&= \mathbb{E}\left[\frac{1}{5}(X-1)\right] && \text{(LOTUS)} \\
&= \frac{1}{5}(\mathbb{E}[X] - 1) && \text{(linearity)}
\end{aligned}$$

Since $X \sim \text{Geometric}(\frac{1}{6})$, the expected number of rolls until we roll a 6 is $\mathbb{E}[X] = 6$:

$$= \frac{1}{5}(6-1) = 1$$

Alternatively, we can use iterated expectation, along with the fact that $\mathbb{E}[Y \mid X] = \frac{1}{5}(X-1)$, to give

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y \mid X]] \\
&= \mathbb{E}\left[\frac{1}{5}(X-1)\right] \\
&= \frac{1}{5}(\mathbb{E}[X]-1) \\
&= \frac{1}{5}(6-1) = 1
\end{aligned}$$

(b) We use the same logic as the first part, except now each of the first $k-1$ rolls can only be 1, 2, or 3, so

$$\mathbb{E}[Y \mid X = k] = \frac{1}{3}(k-1).$$

Using the law of total expectation, we have

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{k=1}^{\infty} \mathbb{E}[Y \mid X = k]\,\mathbb{P}[X = k] && \text{(total expectation)} \\
&= \sum_{k=1}^{\infty} \frac{1}{3}(k-1)\,\mathbb{P}[X = k] \\
&= \mathbb{E}\left[\frac{1}{3}(X-1)\right] && \text{(LOTUS)} \\
&= \frac{1}{3}(\mathbb{E}[X]-1) && \text{(linearity)}
\end{aligned}$$

Since now $X \sim \text{Geometric}(\frac{1}{2})$, the expected number of rolls until we roll a number greater than 3 is $\mathbb{E}[X] = 2$:

$$= \frac{1}{3}(2-1) = \frac{1}{3}$$

Alternatively, we can use iterated expectation, along with the fact that $\mathbb{E}[Y \mid X] = \frac{1}{3}(X-1)$, to give

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\mathbb{E}[Y \mid X]] \\
&= \mathbb{E}\left[\frac{1}{3}(X-1)\right] \\
&= \frac{1}{3}(\mathbb{E}[X]-1) \\
&= \frac{1}{3}(2-1) = \frac{1}{3}
\end{aligned}$$