CS 70 Discrete Mathematics and Probability Theory
Spring 2024 Seshia, Sinclair

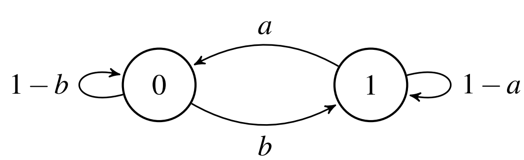
Discussion 13B

1 Markov Chain Terminology

Note 22

In this question, we will walk you through terms related to Markov chains.

1. (Irreducibility) A Markov chain is irreducible if, starting from any state i, the chain can transition to any other state j, possibly in multiple steps.
2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i,i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathscr{X}$. If $d(i) = 1\ \forall i \in \mathscr{X}$, then the Markov chain is aperiodic; otherwise it is periodic.
3. (Matrix Representation) Define the transition probability matrix P by filling entry (i,j) with probability $P(i,j)$.
4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

(d) Construct a transition probability matrix using the above Markov chain.

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

Solution:

(a) The Markov chain is irreducible if both a and b are non-zero. It is reducible if at least one of a and b is 0.

(b) We compute $d(0)$ to find that:

$$d(0) = \gcd\{2, 4, 6, \ldots\} = 2.$$

This is because if we start at a state X then we can get back to it after taking an even number of steps only (2, 4, 6, 8, etc.), not by taking an odd number of steps (1, 3, 5, 7, etc.). Thus, the chain is periodic with period 2.

(c) We compute $d(0)$ to find that:

$$d(0) = \gcd\{1, 2, 3, ...\} = 1.$$

Thus, the chain is aperiodic. Notice that the self-loops allow us to stay at the same node, thereby letting us get to any other node in an odd *or* even number of steps.

(d)

$$\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

(e)

$$\pi(0) = (1-b)\pi(0) + a\pi(1),$$
$$\pi(1) = b\pi(0) + (1-a)\pi(1).$$

One of the equations is redundant. We throw out the second equation and replace it with $\pi(0) + \pi(1) = 1$. This gives the solution

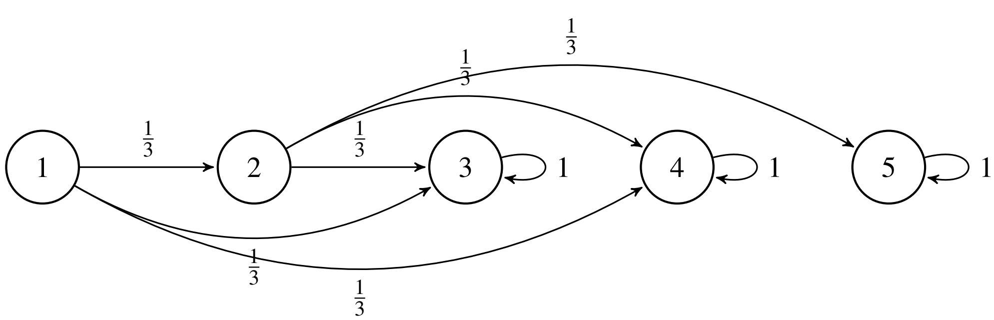
$$\pi = \frac{1}{a+b}\begin{bmatrix} a & b \end{bmatrix}.$$

2 Skipping Stones

Note 22

We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be $\mathscr{X} = \{1, 2, 3, 4, 5\}$. State 3 represents the target, while states 4 and 5 indicate that you have overshot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of $\{3\}$ before $\{4, 5\}$.

Solution: Here is the Markov Chain we are working with:



Let $\alpha(i)$ denote the probability of reaching the target before overshooting, starting at state i. Then:

$$\begin{aligned}
\alpha(5) &= 0 \\
\alpha(4) &= 0 \\
\alpha(3) &= 1 \\
\alpha(2) &= \frac{1}{3}\alpha(3) + \frac{1}{3}\alpha(4) + \frac{1}{3}\alpha(5) = \frac{1}{3} \\
\alpha(1) &= \frac{1}{3}\alpha(2) + \frac{1}{3}\alpha(3) + \frac{1}{3}\alpha(4) = \frac{1}{9} + \frac{1}{3}
\end{aligned}$$

Therefore, $\alpha(1) = 1/9 + 1/3 = 4/9$.

3 Consecutive Flips

Note 22 Suppose you are flipping a fair coin (one Head and one Tail) until you get the same side 3 times (Heads, Heads, Heads) or (Tails, Tails, Tails) in a row.

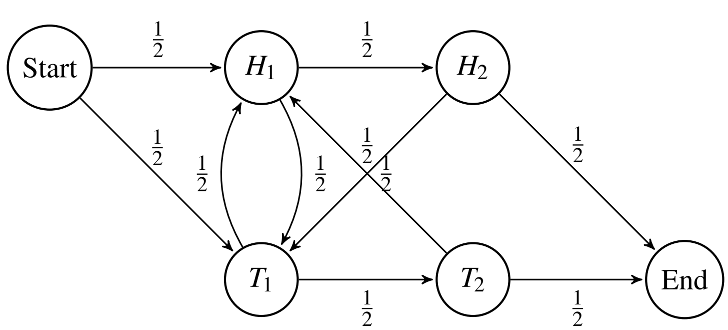
(a) Construct an Markov chain that describes the situation with a start state and end state.

(b) Given that you have flipped a (Tails, Heads) so far, what is the expected number of flips to see the same side three times?

(c) What is the expected number of flips to see the same side three times, beginning at the start state?

Solution:

(a) The appropriate Markov chain has 6 states: Start, H_1, H_2, T_1, T_2, and End.
For starting node, there is an outgoing edge to H_1 and T_1, each with equal probability of $1/2$.
For H_1, there is an outgoing edge to H_2 and T_1, each with equal probability of $1/2$.
For H_2, there is an outgoing edge to End and T_1, each with equal probability of $1/2$.
For T_1, there is an outgoing edge to H_1 and T_2, each with equal probability of $1/2$.
For T_2, there is an outgoing edge to H_1 and End, each with equal probability of $1/2$.



(b) If you got a Tails and then a Heads, you are currently in the H_1 state. Thus, we must calculate the expected number of flips to end from H_1. Thus we will do this with a system of equations. Since we are not trying to solve for the starting state, we have 5 unknowns that depend on 5 linearly independent equations. Let $\beta(i)$ be the expected number of flips to reach the end state starting from state i. Then we have:

$$\begin{aligned}
\beta(H_1) &= 1 + 0.5\beta(T_1) + 0.5\beta(H_2) \\
\beta(H_2) &= 1 + 0.5\beta(\text{End}) + 0.5\beta(T_1) \\
\beta(T_1) &= 1 + 0.5\beta(T_2) + 0.5\beta(H_1) \\
\beta(T_2) &= 1 + 0.5\beta(\text{End}) + 0.5\beta(H_1) \\
\beta(\text{End}) &= 0
\end{aligned}$$

If we solve this system of equations, we get $\beta(H_1) = 6, \beta(H_2) = 4, \beta(T_1) = 6, \beta(T_2) = 4$.

(c) $\beta(S) = 1 + 0.5\beta(H_1) + 0.5\beta(T_1) = 1 + 0.5 \cdot 6 + 0.5 \cdot 6 = 7$.