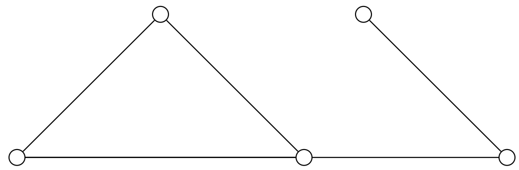
Discussion 2C

CS 70, Summer 2024

1 Degree Sequences

The *degree sequence* of a graph is the sequence of the degrees of the vertices, arranged in descending order, with repetitions as needed. For example, the degree sequence of the following graph is $(3, 2, 2, 2, 1)$.



For each of the parts below, determine whether there exists an undirected graph G with the given degree sequence. Justify your claims.

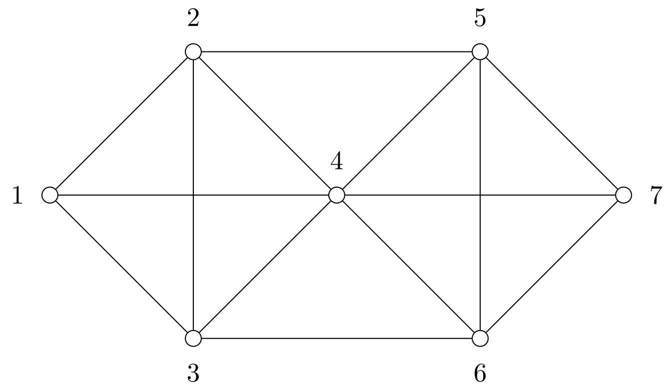
(a) $(3, 3, 2, 2)$.

(b) $(3, 2, 2, 2, 2, 1, 1)$.

(c) $(6, 2, 2, 2)$.

(d) $(4, 4, 3, 2, 1)$.

2 Eulerian Tour and Eulerian Walk



(a) Determine whether there exists an Eulerian tour in the graph above. If there does not, provide justification. Otherwise, provide an example.

(b) An Eulerian walk is a walk that uses each edge exactly once. Determine whether there is an Eulerian walk in the graph above. If there is not, provide justification. Otherwise, provide an example.

(c) Find a sufficient condition for an undirected graph to have an Eulerian walk. Prove your answer.

3 Build-Up Error

Consider the following proof.

Claim. If every vertex in an undirected graph has degree at least one, then the graph is connected.

Proof. By induction on the number of vertices n.

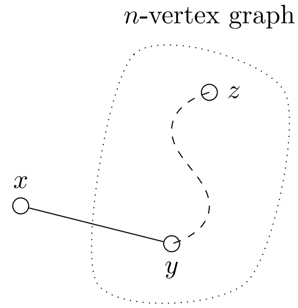
Base case. $n = 1$. This is the graph with only a single vertex. In this case, the hypothesis is false, and thus the claim is vacuously true.

Induction case.

Induction hypothesis. Suppose that for some $n \geq 1$, if every vertex in an undirected graph with n vertices has degree at least one, then the graph is connected.

Induction step. Consider any undirected graph with n vertices, where every vertex has degree at least one. By the induction hypothesis, this graph is connected. Connect a new vertex x to a vertex in this graph to obtain an undirected graph with $n + 1$ vertices, each of which has degree at least one.

It remains to show that this new graph with $n + 1$ vertices is connected. In particular, we must show that there is a path from our vertex x to any other vertex z. Let y be the vertex in the n-vertex graph that x was connected to. Since the n-vertex graph is connected, there is a path from y to z. Thus we can find a path from x to z by adjoining $\{x, y\}$ to our sequence of edges in the path from y to z.



By the principle of mathematical induction, we have shown that every undirected graph with vertices all of degree at least one is connected.

(a) Disprove the claim with a counterexample.

(b) Explain what is wrong with this proof.

(c) The error in this proof is known as "build-up error." Explain how graph induction proofs should be structured to avoid such errors.

4 Odd-Degree Vertices

Consider the following claim: for $G = (V, E)$ an undirected graph, G has an even number of vertices with odd degree.

Prove this claim using each of the below techniques.

(a) Via a direct proof (e.g., counting the number of edges in G).

(*Hint*: use the Handshaking Lemma.)

(b) By induction on $m = |E|$.

(c) By induction on $n = |V|$.