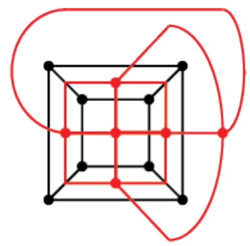
# Discussion 2D

*CS 70, Summer 2024*

## 1 Cube Dual

**(a)** Here is one possible drawing of the cube (in black) with its dual (in red):

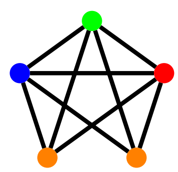


As seen in the drawing, the dual is indeed planar.

**(b)** From the drawing, $G'$ is not bipartite. This is a reminder that connecting the middle of every face on a cube does not result in another cube, which would be bipartite!

## 2 Planarity and Coloring

**(a)** We prove by construction. Consider $K_5$, which has 5 vertices and 10 edges and can be colored with five colors. Remove one edge to get a new graph with 5 vertices and 9 edges. Since the vertices that were connected by this edge are now no longer adjacent, we can now use the same color for these two vertices. This allows us to color the new graph with four colors, as desired.

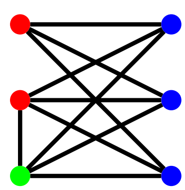


**(b)** To disprove the statement, we prove that any such graph must be planar, and therefore must be four-colorable.

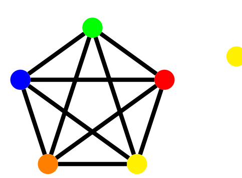
Such a graph cannot a subdivision of $K_5$ as it does not have enough edges. $K_5$ has 10 edges. Similarly, such a graph cannot contain a subdivision of $K_{3,3}$, since it does not have enough vertices. $K_{3,3}$ has 6 vertices.

Therefore, by Kuratowski's theorem, since such a graph does not contain a subdivision of $K_5$ nor a subdivision of $K_{3,3}$, it must be planar. By the four-color theorem, such a graph much be four-colorable.

**(c)** We prove by construction. Consider $K_{3,3}$, which has 6 vertices and 9 edges and can be colored with two colors since it is bipartite. Add an additional edge anywhere to get a new graph with 6 vertices and 10 edges. Such an edge will be between two vertices of the same group, and we can thus use a new color for one of those two vertices. This colors the graph with three colors, so it is certainly four-colorable.



**(d)** We prove by construction. Consider $K_5$, which has 5 vertices and 10 edges and cannot be colored with fewer than five colors since it is fully connected. Add an additional isolated vertex to get a new graph with 6 vertices and 10 edges. This graph cannot be colored with fewer than five colors since we have not removed any edges. So it is not four-colorable.



# 3 Touring Hypercube

**(a)** In the $n$-dimensional hypercube $H_n$, two vertices are connected if they differ in exactly one bit location. Since the vertices are binary strings of length $n$, this means that every vertex has degree $n$.

If $n$ is odd, then by Euler's Theorem there can be no Eulerian tour. On the other hand, the hypercube is connected. Therefore, when $n$ is even, since every vertex has even degree and the graph is connected, there is an Eulerian tour.

**(b)** By induction on $n$, the dimension of the hypercube.

**Base case**. $n = 1$. In the two-dimensional hypercube, there are only two vertices connected by an edge. We can form a Hamiltonian tour by walking from one to the other and then back.

**Induction case**.

**Induction hypothesis**. For $n \geq 2$, suppose that the $n$-dimensional hypercube $H_n$ has a Hamiltonian tour.

**Induction step**. Consider $H_{n+1}$. Let $H_0$ and $H_1$ be the subgraphs of $H_{n+1}$ consisting with vertices with initial bits 0 and 1, respectively. Then $H_0$ and $H_1$ are each $n$-dimensional hypercubes. By the induction hypothesis, they each have a Hamiltonian tour.

We construct a tour in $H_{n+1}$. Let $x_0$ be an arbitrary vertex in $H_0$. Follow the tour in $H_0$ up until the very edge which returns to $x_0$. Let $\{x_0, y_0\}$ be this edge.

Take the edge $\{y_0, y_1\}$ to enter $H_1$. Follow the tour in $H_1$ backwards from $y_1$ to arrive at $x_1$. Finally, take the edge $\{x_1, x_0\}$ to finish the tour.

# 4 Binary Trees

**(a)** **(i)** Since $r$ has degree 2, removing it will break $T$ into two connected components. Let those components be $L$ and $R$.

Before removing $r$, $u$ must have had degree 1 or 3.

(1) If $u$ had degree 1, then after removing $r$, $u$ is a single vertex. This is a binary tree of height 0, and its root is $u$.

(2) If $u$ had degree 3, then after removing $r$, $u$ has degree 2.

All other vertices in $L$ have degree 1 or 3. Therefore $L$ is a binary tree with root $u$.

The same reasoning shows that $R$ would also be a binary tree with root $v$.

**(ii)** Let $u$ and $v$ be the roots of $L$ and $R$ as in part **(a)(i)**. Because $T$ is a tree, any path from $r$ to a leaf must go through either $u$ or $v$, but not both. This means that the maximum distance from $r$ to any leaf is one more than either the maximum distance from $u$ to any leaf in $L$ or the maximum distance from $v$ to any leaf in $R$.

Formally, if we define $\mathcal{L}(L)$ and $\mathcal{L}(R)$ to be the set of leaves in $L$ and $R$ respectively and $d(r, \ell)$ as the length of the path from $r$ to some leaf $\ell$, then we have

$$\begin{aligned} h(T) &= \max_{\ell \in \mathcal{L}(T)} d(r, \ell) \\ &= 1 + \max\left( \max_{\ell \in \mathcal{L}(L)} d(u, \ell), \max_{\ell \in \mathcal{L}(R)} d(v, \ell) \right) \\ &= 1 + \max(h(L), h(R)). \end{aligned}$$

**(b)** We proceed by induction on the height of the binary tree.

**Base case**. $h = 0$. A binary tree of height 0 is a singleton and so has $2^1 - 1 = 1$ vertex.

**Induction case**.

**Induction hypothesis**. Suppose that for all $k < h$, every binary tree of height $h$ has at most $2^{k+1} - 1$ vertices.

**Induction step**. Consider any binary tree $T$ of height $h$. By part **(a)(i)**, we can remove the root from $T$ to obtain two binary trees $L$ and $R$. By part **(a)(ii)**, since $h = \max(h(L), h(R)) + 1$, we have that $h(L) < h$ and $h(R) < h$. So we can apply the induction hypothesis: $L$ has at most $2^{h(L)+1} - 1$ vertices and $R$ has at most $2^{h(R)+1} - 1$ vertices.

The vertices in $T$ consist of the vertices in $L$, the vertices in $R$, and the root. So the number of vertices in $T$ is at most.

$$2^{h(L)+1} - 1 + 2^{h(R)+1} - 1 + 1 \leq 2^{h+1} + 2^{h+1} - 1 = 2^{h+1} - 1.$$

By the principle of mathematical induction, we have shown that $T$ has at most $2^{h+1} - 1$ vertices.

**(c)** By induction on $n$.

**Base case**. $n = 1$. If a binary tree has one leaf, it is the graph with only one vertex. It has $1 = 2 \cdot 1 - 1$ vertices.

**Induction case**.

**Induction hypothesis**. Suppose that for all $k < n$, every binary tree with $k$ leaves has $2k - 1$ vertices.

**Induction step**. Let $T$ be any binary tree with $n > 1$ leaves. Remove the root $r$ to break $T$ into two binary trees $L$ and $R$. Note that since $n > 1$, the root itself is not a leaf.

Let $a$ and $b$ be the numbers of leaves in $L$ and $R$, respectively. Note that all the leaves of $T$ are in $L$ or $R$ since the root is not a leaf. So $a + b = n$. By the induction hypothesis, $L$ has $2a - 1$ vertices and $R$ has $2b - 1$ vertices. So the number of vertices in $T$ is

$$(2a - 1) + (2b - 1) + 1 = 2(a + b) - 1 = 2n - 1.$$

By the principle of mathematical induction we have shown that every binary tree with $n$ leaves has $2n - 1$ vertices.