CS 70 Discrete Mathematics and Probability Theory
Summer 2025 Tate

HW 00

1 Administrivia

(a) Make sure you are on the course Ed (for Q&A) and Gradescope (for submitting homeworks, including this one). Find and familiarize yourself with the course website. What is its homepage's URL?

(b) Read the policies page on the course website.

(i) What is the breakdown of how your grade is calculated?

(ii) What is the attendance policy for discussions?

(iii) When are homeworks released, and when are they due?

(iv) How many "drops" do you get for homeworks? How many mini-vitamins will contribute to your grade?

(v) When is the midterm? When is the final?

(vi) What percentage score is needed to earn full credit on a homework?

Solution:

(a) The course website is located at `https://www.eecs70.org/`.

(b) (i) Discussion Attendance: 5%, Mini-vitamins: 5%, Homework: 15%, Midterm: 30%, Final: 45%.

(ii) You will receive 1 attendance point for every discussion, and will need at least 13 points in order to receive full credit for discussion attendance. You are welcome to attend other discussion sections, and may receive credit for any section, though it is *strongly encouraged* that you consistently attend the discussion that you sign up for at the beginning of the semester.

(iii) The homework for the current week is released on the course website on Saturday. The homework is due on Gradescope the following Saturday at 4:00 PM (grace period until 6:00 PM); the solutions for that homework will be released on Monday, a day after the release of the new homework.

(iv) You can drop the lowest 2 homeworks the entire semester, and the top 13 mini vitamins will count for a grade. However, please save these drops for emergencies. We do not have the bandwidth to make personalized exceptions to this rule.

(v) Midterm Date: 7/15/25 Tuesday 7–9pm Final Date: 8/12/25 Tuesday 7–10pm

(vi) 73%.

2 Course Policies

Go to the course website and read the course policies carefully. Leave a followup on Ed if you have any questions. Are the following situations violations of course policy? Write "Yes" or "No", and a short explanation for each.

(a) Alice and Bob work on a problem in a study group. They write up a solution together and submit it, noting on their submissions that they wrote up their homework answers together.

(b) Carol goes to a homework party and listens to Dan describe his approach to a problem on the board, taking notes in the process. She writes up her homework submission from her notes, crediting Dan.

(c) Erin comes across a proof that is part of a homework problem while studying course material. She reads it and then, after she has understood it, writes her own solution using the same approach. She submits the homework with a citation to the website.

(d) Frank is having trouble with his homework and asks Grace for help. Grace lets Frank look at her written solution. Frank copies it onto his notebook and uses the copy to write and submit his homework, crediting Grace.

(e) Heidi has completed her homework using LaTeX. Her friend Irene has been working on a homework problem for hours, and asks Heidi for help. Heidi sends Irene her PDF solution, and Irene uses it to write her own solution with a citation to Heidi.

(f) Joe found homework solutions before they were officially released, and every time he got stuck, he looked at the solutions for a hint. He then cited the solutions as part of his submission.

(g) Kai is struggling with one of their homework problems, so they take a screenshot of the problem and ask ChatGPT to solve it for them. They adapt ChatGPT's response to their own solution, and they include a link to their ChatGPT conversation in the homework's Sundry section.

Solution:

(a) Yes, this is a violation of course policy. All solutions must be written entirely by the student submitting the homework. Even if students collaborate, each student must write a unique, individual solution. In this case, both Alice and Bob would be culpable.

(b) No, this is not a violation of course policy. While sharing *written solutions* is not allowed, sharing *approaches* to problems is allowed and encouraged. Because Carol only copied down *notes*, not *Dan's solution*, and properly cited Dan's contribution, this is an actively encouraged form of collaboration.

(c) No, this is not a violation of course policy. Using external sources to help with homework problems, while less encouraged than peer collaboration, is fine as long as (i) the student makes sure to understand the solution; (ii) the student uses understanding to write a new solution, and does not copy from the external source; and (iii) the student credits the external source. However, looking up a homework problem online is a violation of course policies; the correct course of action upon finding homework solutions online is to close the tab.

(d) Yes, this is a violation of course policy, and both Frank and Grace would be culpable. Even though Frank credits Grace, written solutions should never be shared in the first place, and certainly not copied down. This is to ensure that each student learns how to write and present clear and convincing arguments. To be safe, try not to let anybody see your written solutions at any point in the course—restrict your collaboration to *approaches* and *verbal communication*.

(e) Yes, this is a violation of course policy. Once again, a citation does not make up for the fact that written solutions should never be shared, in written or typed form. In this case, both Heidi and Irene would be culpable.

(f) Yes, this is a violation of course policy. Joe should not be reading solutions before they are officially released. Instead, Joe should ask for help when he is stuck through Ed or Office Hours.

(g) Yes, this is a violation of course policy. Kai should not be directly asking ChatGPT to solve a homework question, despite correctly including the conversation history in their Sundry. If Kai really wanted to use an LLM for homework help, they should ask it a conceptual question (that they'd reasonably ask at Office Hours or on Ed).

3 Use of Ed

Ed is incredibly useful for Q&A in such a large-scale class. We will use Ed for all important announcements. You should check it frequently. We also highly encourage you to use Ed to ask questions and answer questions from your fellow students.

(a) Read the Ed Etiquette section of the course policies and explain what is wrong with the following hypothetical student question: "Can someone explain the proof of Theorem XYZ to me?" (Assume Theorem XYZ is a complicated concept.)

(b) When are the weekly posts released? Are they required reading?

(c) If you have a question or concern not directly related to the course content, where should you direct it?

Solution:

(a) There are two things wrong with this question. First, this question does not pass the 5-minute test. This concept takes longer than 5 minutes to explain, and therefore is better suited to Office Hours. Second, this question does not hone in on a particular concept with which the student is struggling. Questions on Ed should be narrow, and should include every step of reasoning that led up to the question. A better question in this case might be: "I understood every step of the proof of Theorem XYZ in Note 2, except for the very last step. I tried to reason it like this, but I didn't see how it yielded the result. Can someone explain where I went wrong?"

(b) The weekly posts are released every Monday. They're required reading.

(c) Please send an email to `cs70-staff@berkeley.edu`.

4 Academic Integrity

Please write or type out the following pledge in print, and sign it.

> I pledge to uphold the university's honor code: to act with honesty, integrity, and respect for others, including their work. By signing, I ensure that all written homework I submit will be in my own words, that I will acknowledge any collaboration or help received, and that I will neither give nor receive help on any examinations.

5 Propositional Practice

Note 1

In parts (a)–(b), convert the English sentences into propositional logic. In parts (c)–(d), convert the propositions into English. For parts (b) and (d), use the notation $a \mid b$ to denote the statement "a divides b", and use the notation $P(x)$ to denote the statement "x is a prime number".

(a) For every real number k, there is a unique real solution to $x^3 = k$.

(b) If p is a prime number, then for any two natural numbers a and b, if p doesn't divide a and p divides ab, then p divides b.

(c) $(\forall x, y \in \mathbb{R})[(xy = 0) \implies ((x = 0) \vee (y = 0))]$

(d) $\neg((\exists y \in \mathbb{N})[(\forall x \in \mathbb{N})[(x > y) \implies ((y \mid x) \vee P(x))]])$

Solution:

(a) The trickiest part of this problem is the word 'unique'. We can express the existence of a unique solution in propositional logic with two statements connected with an 'and': (1) A solution exists, and (2) Any two solutions have to be the same. Hence, we can rewrite this statement as "For every real number k, there exists a real number x such that $x^3 = k$ and for all reals y and z, if both $y^3 = k$ and $z^3 = k$, then $y = z$." This, in propositional logic, is below:

$$(\forall k \in \mathbb{R})\big[(\exists x \in \mathbb{R})(x^3 = k) \wedge (\forall y, z \in \mathbb{R})(((y^3 = k) \wedge (z^3 = k)) \implies (y = z))\big].$$

(b) This sentence can be written in propositional logic as

$$(\forall p \in \mathbb{N})[(P(p)) \implies ((\forall a, b \in \mathbb{N})[((p \mid ab) \wedge \neg(p \mid a)) \implies (p \mid b)])].$$

(c) If the product of two real numbers is 0, then one of them must be 0.

(d) There is no natural number that divides every composite number greater than it.

6 More Logical Equivalences

Note 1 Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a) $\forall x(P(x) \implies Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \implies \forall x\, Q(x)$

(b) $\neg(\exists x(P(x) \vee Q(x))) \stackrel{?}{\equiv} \forall x(\neg P(x) \wedge \neg Q(x))$

(c) $\forall x((P(x) \implies Q(x)) \wedge Q(x)) \stackrel{?}{\equiv} \forall x\, P(x)$

Solution:

(a) Not Equivalent.

Justification: Consider the case where $P(x)$ can be true or false depending on x (e.g. $P(x) = x > 0$) and $Q(x)$ is always false; (e.g. $Q(x) = x \notin \mathbb{R}$), where the universe is the real numbers.

The LHS is false because there are values of x such that the implication $P(x) \implies Q(x)$ is false (e.g. $x = 1$). However, the RHS is vacuously true because $P(x)$ is not always true.

(b) Equivalent.

Justification:

$$\begin{aligned} \neg(\exists x(P(x) \vee Q(x))) &\equiv \forall x(\neg(P(x) \vee Q(x))) \\ &\equiv \forall x(\neg P(x) \wedge \neg Q(x)) \qquad \text{[De Morgan's Law]} \end{aligned}$$

Intuitively, the LHS says that there is no x such that $P(x) \vee Q(x)$, so $P(x) \vee Q(x)$ is false everywhere. This means that $\neg(P(x) \vee Q(x))$ must be true everywhere, so by De Morgan's $\neg P(x) \wedge \neg Q(x)$ is true everywhere, which is the RHS.

(c) Not Equivalent.

Justification: Consider the case where $Q(x)$ is always true; for example, $Q(x) = x \in \mathbb{R}$, where the universe is the real numbers. Then regardless of the truth value of $P(x)$, $P(x) \implies Q(x)$ would be true, and thus the LHS would always be true.

Then we can choose some $P(x)$ which is not always true - e.g. $P(x) = x > 0$, so the RHS would be false, and the two sides are not equivalent.

7 Prove or Disprove

Note 2 For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$, if r is irrational, then $r+1$ is irrational.

(d) $(\forall n \in \mathbb{Z}^+)\ 5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

(e) The product of a non-zero rational number and an irrational number is irrational.

(f) If $A \subseteq B$, then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. (Recall that $A' \in \mathscr{P}(A)$ if and only if $A' \subseteq A$.)

Solution:

(a) **Answer**: True.

Proof. We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k+1$ for some natural number k. This means that we have

$$\begin{aligned} n^2 + 4n &= (2k+1)^2 + 4(2k+1) \\ &= 4k^2 + 12k + 5 \\ &= 2(2k^2 + 6k + 2) + 1 \end{aligned}$$

Since $2k^2 + 6k + 2$ is a natural number, by the definition of odd numbers, $n^2 + 4n$ is odd.

Alternatively, we could also factor the expression to get $n(n+4)$. Since n is odd, $n+4$ is also odd. The product of 2 odd numbers is also an odd number. Hence n^2+4n is odd. □

(b) **Answer**: True.

Proof. We will use a proof by contraposition. Suppose that $a > 11$ and $b > 4$ (note that this is equivalent to $\neg(a \leq 11 \vee b \leq 4)$). Since $a > 11$ and $b > 4$, $a+b > 15$ (note that $a+b > 15$ is equivalent to $\neg\big(a+b \leq 15\big)$). Thus, if $a+b \leq 15$, then $a \leq 11$ or $b \leq 4$. □

(c) **Answer**: True.

Proof. We will use a proof by contraposition. Assume that $r+1$ is rational. Since $r+1$ is rational, it can be written in the form $\frac{a}{b}$ where a and b are integers. Then r can be written as $\frac{a-b}{b}$. By the definition of rational numbers, r is a rational number, since both $a-b$ and b are integers. By contraposition, if r is irrational, then $r+1$ is irrational. □

(d) **Answer**: False.

Proof. We will show a counterexample. Let $n = 7$. Here, $5 \cdot 7^3 = 1715$, but $7! = 5040$. Since $5n^3 < n!$, the claim is false.

A counterexample that is easier to see without much calculation is for a much larger number like $n = 100$; here, $100!$ is clearly more than $5 \cdot 100^3 = 100 \cdot 50 \cdot 25 \cdot 5 \cdot 4 \cdot 2$, since the latter product contains only a subset of the terms in $100!$. □

(e) **Answer**: True.

Proof. We prove the statement by contradiction. Suppose that $ab = c$, where $a \neq 0$ is rational, b is irrational, and c is rational. Since a and b are not zero (because 0 is rational), c is also non-zero. Thus, we can express $a = \frac{p}{q}$ and $c = \frac{r}{s}$, where p, q, r, and s are nonzero integers. Then

$$b = \frac{c}{a} = \frac{rq}{ps},$$

which is the ratio of two nonzero integers, giving that b is rational. This contradicts our initial assumption, so we conclude that the product of a nonzero rational number and an irrational number is irrational. □

(f) **Answer**: True.

Proof. Suppose $A' \in \mathscr{P}(A)$; this means that $A' \subseteq A$ (by the definition of the power set).

Let $x \in A'$. Then, since $A' \subseteq A$, $x \in A$. Since $A \subseteq B$, $x \in B$. We have shown $(\forall x \in A')(x \in B)$, so $A' \subseteq B$.

Since the previous argument works for any $A' \subseteq A$, we have proven $(\forall A' \in \mathscr{P}(A))(A' \subseteq B)$. So, $(\forall A' \in \mathscr{P}(A))(A' \in \mathscr{P}(B))$ Thus, we conclude $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ as desired. □

8 Airport

Note 3 Suppose that there are $2n+1$ airports, where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

Solution: We proceed by induction on n. For $n = 1$, let the 3 airports be A, B, C and without loss of generality suppose B, C is the closest pair of airports (which is well defined since all distances are different). Then the airplanes departing from B and C are flying towards each other. Since the airplane from A must fly to somewhere else, no airplanes are destined for airport A.

Now suppose the statement holds for $n = k$, i.e. when there are $2k+1$ airports. For $n = k+1$, i.e. when there are $2k+3$ airports, the airplanes departing from the closest two airports (say X and Y) must be destined for each other's starting airports. Removing these two airports reduce the problem to $2k+1$ airports.

From the inductive hypothesis, we know that among the $2k+1$ airports remaining, there is an airport with no incoming flights which we call airport Z. When we add back the two airports that we removed, there are two scenarios:

- Some of the flights get remapped to X or Y.
- None of the flights get remapped.

In either scenario, we conclude that the airport Z will continue to have no incoming flights when we add back the two airports, and so the statement holds for $n = k+1$. By induction, the claim holds for all $n \geq 1$.