CS 70 Discrete Mathematics and Probability Theory — Summer 2025 Tate — HW 01

Due: Saturday, 7/5, 4:00 PM
Grace period until Saturday, 7/5, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.) If you used an LLM, place transcripts of your chats here.

1 Universal Preference

Note 4

Suppose that preferences in a stable matching instance are universal: all n jobs share the preferences $C_1 > C_2 > \cdots > C_n$ and all candidates share the preferences $J_1 > J_2 > \cdots > J_n$.

(a) What pairing do we get from running the algorithm with jobs proposing? Prove that this happens for all n.

(b) What pairing do we get from running the algorithm with candidates proposing? Explain.

(c) What does this tell us about the number of stable pairings? Justify your answer.

2 Pairing Up

Note 4

Prove that for every even $n \geq 2$, there exists an instance of the stable matching problem with n jobs and n candidates such that the instance has at least $2^{n/2}$ distinct stable matchings.

(*Hint*: It can help to start with some small examples; find an instance for $n = 2$, and think about how you can use these preference lists to construct an instance for $n = 4$. After this, you should be in a good position to generalize the construction for all even n.)

3 Short Tree Proofs

Note 5

Let $G = (V, E)$ be an undirected graph with $|V| \geq 1$.

(a) Prove that every connected component in an acyclic graph is a tree.

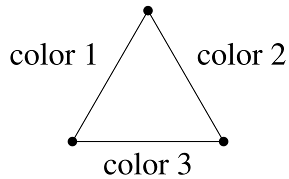
(b) Suppose G has k connected components. Prove that if G is acyclic, then $|E| = |V| - k$.

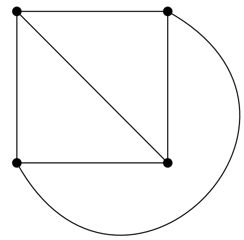
(c) Prove that a graph with $|V|$ edges contains a cycle.

4 Edge Colorings

Note 5

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.





(a) Show that the 4 vertex complete graph above can be 3 edge colored. (You may use the numbers $1,2,3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d-1$ colors.

(c) Prove that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

5 Touring Hypercube

Note 5

In the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph (with ≥ 2 vertices) is a tour that visits every vertex exactly once.

(a) Prove that a hypercube has an Eulerian tour if and only if n is even.

(b) Prove that every hypercube has a Hamiltonian tour.

6 Planarity and Graph Complements

Note 5

Let $G=(V,E)$ be an undirected graph. We define the complement of G as $\overline{G}=(V,\overline{E})$ where $\overline{E}=\{(i,j) \mid i,j \in V, i \neq j\}-E$; that is, $\overline{G}$ has the same set of vertices as G, but an edge e exists is $\overline{G}$ if and only if it does not exist in G.

(a) Suppose G has v vertices and e edges. How many edges does $\overline{G}$ have?

(b) Prove that for any graph with at least 13 vertices, G being planar implies that $\overline{G}$ is non-planar.

(c) Now consider the converse of the previous part, i.e., for any graph G with at least 13 vertices, if $\overline{G}$ is non-planar, then G is planar. Construct a counterexample to show that the converse does not hold.

Hint: Recall that if a graph contains a copy of K_5, then it is non-planar. Can this fact be used to construct a counterexample?

7 Modular Practice

Note 6

(a) Compute $13^{2025} \pmod{12}$.

(b) Compute $7^{62} \pmod{11}$.

(c) Solve for x: $10x + 5 \equiv 7 \pmod{13}$.

(d) Solve the system of equations for x and y: $5x + 4y \equiv 0 \pmod 7$ and $2x + y \equiv 4 \pmod 7$.

(e) Prove that $3x + 12 \equiv 4 \pmod{21}$ does not have a solution.

8 How Many Solutions?

Note 6

Consider the equation $ax \equiv b \pmod p$ for prime p. In the below three parts, all values a, b, x are defined as values in the range $\{0, 1, \ldots p-1\}$. In addition, include justification for your answers to all the subparts of this problem.

(a) For how many pairs (a, b) does the equation have a unique solution?

(b) For how many pairs (a, b) does the equation have no solution?

(c) For how many pairs (a, b) does the equation have p solutions?

Now, consider the equation $ax \equiv b \pmod{pq}$ for distinct primes p, q. In the below three parts, all values a, b, x are defined as values in in the range $\{0, 1, \ldots pq-1\}$.

(d) If $\gcd(a, pq) = p$, show that there exists a solution if and only if $b = 0 \pmod p$. (Hint: Try to relate modular equations to their corresponding algebraic equations, and vice versa.)

(e) If $\gcd(a, pq) = p$ and there is a solution x, show that there are exactly p solutions. (Hint: consider how you can generate another solution $x + \_\_\_$)

(f) For how many pairs (a, b) are there exactly p solutions?