CS 70 Fall 2024 — Discrete Mathematics and Probability Theory — Rao, Hug — HW 03

1 Short Tree Proofs

Note 5

Let $G = (V, E)$ be an undirected graph with $|V| \geq 1$.

(a) Prove that every connected component in an acyclic graph is a tree.

(b) Suppose G has k connected components. Prove that if G is acyclic, then $|E| = |V| - k$.

(c) Prove that a graph with $|V|$ edges contains a cycle.

Solution:

(a) Every connected component is connected, and acyclic because the graph is acyclic; by definition, this is a tree.

(b) Because each connected component is a tree, each connected component has $|V_i| - 1$ edges. The total number of edges is thus $\sum_i (|V_i| - 1) = |V| - k$.

(c) An acyclic graph has $|V| - k$ edges which cannot equal $|V|$, thus if a graph has $|V|$ edges it has a cycle.

2 Proofs in Graphs

Note 5

(a) On the axis from San Francisco traffic habits to Los Angeles traffic habits, Old California is more towards San Francisco: that is, civilized. In Old California, all roads were one way streets. Suppose Old California had n cities ($n \geq 2$) such that for every pair of cities X and Y, either X had a road to Y or Y had a road to X.

Prove that there existed a city which was reachable from every other city by traveling through at most 2 roads.

[*Hint:* Induction]

(b) Consider a connected graph G with n vertices which has exactly $2m$ vertices of odd degree, where $m > 0$. Prove that there are m walks that *together* cover all the edges of G (i.e., each edge of G occurs in exactly one of the m walks, and each of the walks should not contain any particular edge more than once).

[*Hint:* In lecture, we have shown that a connected undirected graph has an Eulerian tour if and only if every vertex has even degree. This fact may be useful in the proof.]

(c) Prove that any graph G is bipartite if and only if it has no tours of odd length.

[*Hint:* In one of the directions, consider the lengths of paths starting from a given vertex.]

Solution:

(a) We prove this by induction on the number of cities n.

Base case: For $n = 2$, there's always a road from one city to the other.

Inductive Hypothesis: When there are k cities, there exists a city c that is reachable from every other city by traveling through at most 2 roads.

Inductive Step: Consider the case where there are $k+1$ cities. Remove one of the cities d and all of the roads to and from d. Now there are k cities, and by our inductive hypothesis, there exists some city c which is reachable from every other city by traveling through at most 2 roads. Let A be the set of cities with a road to c, and B be the set of cities two roads away from c. The inductive hypothesis states that the set S of the k cities consists of $S = \{c\} \cup A \cup B$.

Now add back d and all roads to and from d.

Between d and every city in S, there must be a road from one to the other. If there is at least one road from d to $\{c\} \cup A$, c would still be reachable from d with at most 2 road traversals. Otherwise, if all roads from $\{c\} \cup A$ point to d, d will be reachable from every city in B with at most 2 road traversals, because every city in B can take one road to go to a city in A, then take one more road to go to d. In either case there exists a city in the new set of $k+1$ cities that is reachable from every other city by traveling at most 2 roads.

Alternate Solution :Alternatively, we can prove this using properties of directed graphs. Let c be the city with the largest in-degree. Note that this graph is essentially a complete graph, where each edge is a directed edge instead of an undirected edge. Therefore, the total in degree sums to $n(n-1)/2$, and so does the total out degree. In addition, the in degree + out degree of any vertex must add up to $n-1$.

Because the total in-degree of all vertices is $n(n-1)/2$, The largest in-degree is $d \geq (n-1)/2$. Let S be the these d cities that can reach c by one edge.

For any other city x, it has to have at least $(n-1)-d$ out-degree (because in-degree $<= d$). Notice that there are n total vertices, two of which are x or c, and d vertices that connect to c through one edge. Thus, there are $n-2-d$ other vertices. Since x has out degree at least $n-1-d > n-2-d$, it must therefore connect to at least one vertex in S by the pigeonhole principle.

Thus, all vertices are either connected to c through 1 or 2 edges.

(b) We split the $2m$ odd-degree vertices into m pairs, and join each pair with an edge, adding m more edges in total. (Here, we allow for the possibility of multi-edges, that is, pairs of vertices with more than one edge between them.) Notice that now all vertices in this graph are of even degree. Now by Euler's theorem the resulting graph has an Eulerian tour. Removing the m added edges breaks the tour into m walks covering all the edges in the original graph, with each edge belonging to exactly one walk.

(c) To prove the claim, we need to prove two directions: if G is bipartite, it contains no tours of odd length, and if G contains no tours of odd length, it must be bipartite.

Suppose G is bipartite, and let L and R be the two disjoint sets of vertices such that there does not exist any edge between two vertices in L or two vertices in R. Further, suppose there is some tour in G, and we start traversing this tour at $v_0 \in L$.

Since each edge in G connects a vertex in L to a vertex in R, the first edge in the tour connects the start vertex v_0 to a vertex $v_1 \in R$. Similarly, the second edge connects $v_1 \in R$ to $v_2 \in L$. In general, it must be the case that the $2k$th edge connects vertex $v_{2k-1} \in R$ to $v_{2k} \in L$, and the $2k+1$th edge connects vertex $v_{2k} \in L$ to $v_{2k+1} \in R$.

Since only even numbered edges connect to vertices in L, and we started our tour in L, the tour must end with an even number of edges.

For the opposite direction, suppose G contains no tours of odd length. Without loss of generality, let us consider one connected component of G; the following reasoning can be applied to all of the connected components of G.

Let v be an arbitrary vertex in G; we can divide all of the vertices in G into two disjoint sets:

$$R = \{u \mid \text{the shortest path from } u \text{ to } v \text{ is even}\}$$
$$L = \{u \mid \text{the shortest path from } u \text{ to } v \text{ is odd}\}$$

We claim that no two vertices in L are adjacent. For contradiction, suppose there do exist adjacent vertices $u_1, u_2 \in L$. Consider the tour consisting of:

- the shortest path from v to u_1 (odd length)
- the edge (u_1, u_2) (length 1)
- the shortest path from u_2 to v (odd length)

This tour has odd length, and contradicts our assumption that G has no tours of odd length. This means that no two vertices in L are adjacent.

Similarly, we claim that no two vertices in R are adjacent. For contradiction, suppose there do exist adjacent vertices $u_1, u_2 \in R$. Consider the tour consisting of:

- the shortest path from v to u_1 (even length)
- the edge (u_1, u_2) (length 1)
- the shortest path from u_2 to v (even length)

This tour has odd length, and contradicts our assumption that G has no tours of odd length. This means that no two vertices in R are adjacent.

We've just shown that there are no edges between two vertices in L, and no edges between two vertices in R. If there are multiple connected components in G, the same partition can be applied to all of the components. Together, this means that G is bipartite.

3 Touring Hypercube

Note 5

In the lecture, you have seen that if G is a hypercube of dimension n, then

- The vertices of G are the binary strings of length n.
- u and v are connected by an edge if they differ in exactly one bit location.

A *Hamiltonian tour* of a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that:

- Each vertex appears exactly once in the sequence.
- Each pair of consecutive vertices is connected by an edge.
- v_0 and v_k are connected by an edge.

(a) Show that a hypercube has an Eulerian tour if and only if n is even.

(b) Show that every hypercube has a Hamiltonian tour.

Solution:

(a) In the n-dimensional hypercube, every vertex has degree n. If n is odd, then by Euler's Theorem there can be no Eulerian tour. On the other hand, the hypercube is connected: we can get from any one bit-string x to any other y by flipping the bits they differ in one at a time. Therefore, when n is even, since every vertex has even degree and the graph is connected, there is an Eulerian tour.

(b) By induction on n. When $n = 1$, there are two vertices connected by an edge; we can form a Hamiltonian tour by walking from one to the other and then back.

Let $n \geq 1$ and suppose the n-dimensional hypercube has a Hamiltonian tour. Let H be the $n+1$-dimensional hypercube, and let H_b be the n-dimensional subcube consisting of those strings with initial bit b.

By the inductive hypothesis, there is some Hamiltonian tour T on the n-dimensional hypercube. Now consider the following tour in H. Start at an arbitrary vertex x_0 in H_0, and follow the tour T except for the very last step to vertex y_0 (so that the next step would bring us back to x_0). Next take the edge from y_0 to y_1 to enter cube H_1. Next, follow the tour T in H_1 backwards from y_1, except the very last step, to arrive at x_1. Finally, take the step from x_1 to x_0 to complete the tour. By assumption, the tour T visits each vertex in each subcube exactly once, so our complete tour visits each vertex in the whole cube exactly once.

To build some intuition, here are the first few cases:

- $n = 1$: 0, 1

- $n = 2$: 00, 01, 11, 10

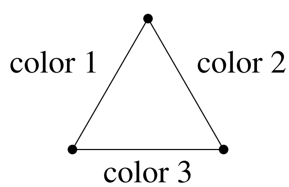
  [Take the $n = 1$ tour in the 0-subcube (vertices with a 0 in front), move to the 1-subcube (vertices with 1 in front), then take the tour backwards. We know 10 connects to 00 to complete the tour.]
- $n = 3$: 000, 001, 011, 010, 110, 111, 101, 100

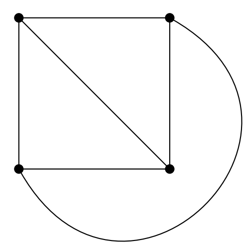
  [Take the $n = 2$ tour in the 0-subcube, move to the 1-subcube, then take the tour backwards. We know 100 connects to 000 to complete the tour.]

The sequence produced with this method is known as a Gray code.

4 Edge Colorings

Note 5 An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.





(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1, 2, 3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d - 1$ colors.

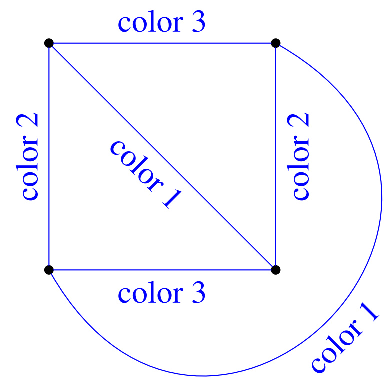
(c) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Solution:

(a) Three color a triangle u_1, u_2, u_3 where (u_1, u_2) is colored 1, (u_2, u_3) is colored 2, and (u_3, u_1) is colored 3. This is a valid 3 coloring as the edges are all colored differently.

Consider adding a fourth vertex v, the incident edges must be colored differently and each incident edge (v, u_i) needs to be colored differently from the edges incident to u_i. That is, one can color (v, u_1) with 2 as it is not incident to the edge colored 2 and that color is avalaible. Similarly one can color edge (v, u_2) with color 3 and (v, u_3) with color 1.

Another proof is simply provide a coloring which is below.



(b) We will use induction on the number of edges n in the graph to prove the statement: If a graph G has $n \geq 0$ edges and the maximum degree of any vertex is d, then G can be colored with $2d-1$ colors.

Base case ($n=0$). If there are no edges in the graph, then there is nothing to be colored and the statement holds trivially.

Inductive hypothesis. Suppose for $n = k \geq 0$, the statement holds.

Inductive step. Consider a graph G with $n = k+1$ edges. Remove an edge of your choice, say e from G. Note that in the resulting graph the maximum degree of any vertex is $d' \leq d$. By the inductive hypothesis, we can color this graph using $2d'-1$ colors and hence with $2d-1$ colors too. The removed edge is incident to two vertices each of which is incident to at most $d-1$ other edges, and thus at most $2(d-1) = 2d-2$ colors are unavailable for edge e. Thus, we can color edge e without any conflicts. This proves the statement for $n = k+1$ and hence by induction we get that the statement holds for all $n \geq 0$.

(c) We will use induction on the number of vertices n in the tree to prove the statement: For a tree with $n \geq 1$ vertices, if the maximum degree of any vertex is d, then the tree can be colored with d colors.

Base case ($n=1$). If there is only one vertex, then there are no edges to color, and thus can be colored with 0 colors.

Inductive hypothesis. Suppose the statement holds for $n = k \geq 1$.

Inductive Step. Remove any leaf v of your choice from the tree. We can then color the remaining tree with d colors by the inductive hypothesis. For any neighboring vertex u of vertex v, the degree of u is at most $d-1$ since we removed the edge $\{u,v\}$ along with the vertex v. Thus its incident edges use at most $d-1$ colors and there is a color available for coloring the edge $\{u,v\}$. This completes the inductive step and by induction we have that the statement holds for all $n \geq 1$.

5 Planarity and Graph Complements

Note 5

Let $G = (V,E)$ be an undirected graph. We define the complement of G as $\overline{G} = (V,\overline{E})$ where $\overline{E} = \{(i,j) \mid i,j \in V, i \neq j\} - E$; that is, $\overline{G}$ has the same set of vertices as G, but an edge e exists is

$\overline{G}$ if and only if it does not exist in G.

(a) Suppose G has v vertices and e edges. How many edges does $\overline{G}$ have?

(b) Prove that for any graph with at least 13 vertices, G being planar implies that $\overline{G}$ is non-planar.

(c) Now consider the converse of the previous part, i.e., for any graph G with at least 13 vertices, if $\overline{G}$ is non-planar, then G is planar. Construct a counterexample to show that the converse does not hold.

Hint: Recall that if a graph contains a copy of K_5, then it is non-planar. Can this fact be used to construct a counterexample?

Solution:

(a) If G has v vertices, then there are a total of $\frac{v(v-1)}{2}$ edges that could possibly exist in the graph. Since e of them appear in G, we know that the remaining $\frac{v(v-1)}{2} - e$ must appear in $\overline{G}$.

(b) Since G is planar, we know that $e \leq 3v - 6$. Plugging this in to the answer from the previous part, we have that $\overline{G}$ has at least $\frac{v(v-1)}{2} - (3v-6)$ edges. Since v is at least 13, we have that $\frac{v(v-1)}{2} \geq \frac{v \cdot 12}{2} = 6v$, so $\overline{G}$ has at least $6v - 3v + 6 = 3v + 6$ edges. Since this is strictly more than the $3v - 6$ edges allowed in a planar graph, we have that $\overline{G}$ must not be planar.

(c) The converse is not necessarily true. As a counterexample, suppose that G has exactly 13 vertices, of which five are all connected to each other and the remaining ten have no edges incident to them. This means that G is non-planar, since it contains a copy of K_5. However, $\overline{G}$ also contains a copy of K_5 (take any 5 of the 8 vertices that were isolated in G), so $\overline{G}$ is also non-planar. Thus, it is possible for both G and $\overline{G}$ to be non-planar.