CS 70 Fall 2024 — Discrete Mathematics and Probability Theory — Rao, Hug — HW 04

1 Modular Practice

Note 6

Solve the following modular arithmetic equations for x and y. For each subpart, show your work and justify your answers.

(a) $9x+5 \equiv 7 \pmod{13}$.

(b) Show that $3x+12 \equiv 4 \pmod{21}$ does not have a solution.

(c) The system of simultaneous equations $5x+4y \equiv 0 \pmod{7}$ and $2x+y \equiv 4 \pmod{7}$.

(d) $13^{2023} \equiv x \pmod{12}$.

(e) $7^{62} \equiv x \pmod{11}$.

Solution:

(a) Subtract 5 from both sides to get:

$$9x \equiv 2 \pmod{13}.$$

Now since $\gcd(9,13)=1$, 9 has a (unique) inverse mod 13, and since $9\times 3 = 27 \equiv 1 \pmod{13}$ the inverse is 3. So multiply both sides by $9^{-1} \equiv 3 \pmod{13}$ to get:

$$x \equiv 6 \pmod{13}.$$

(b) Notice that any number $y \equiv 4 \pmod{21}$ can be written as $y = 4+21k$ (for some integer k). Evaluating y mod 3, we get $y \equiv 1 \pmod{3}$.

Since the right side of the equation is $1 \pmod{3}$, the left side must be as well. However, $3x+12$ will never be $1 \pmod{3}$ for any value of x. Thus, there is no possible solution.

(c) First, subtract the first equation from four times the second equation to get:

$$\begin{aligned} 4(2x+y)-(5x+4y) &\equiv 4(4)-0 \pmod{7} \\ 8x+4y-5x-4y &\equiv 16 \pmod{7} \\ 3x &\equiv 2 \pmod{7} \end{aligned}$$

Multiplying by $3^{-1} \equiv 5 \pmod{7}$, we have $x \equiv 10 \equiv 3 \pmod{7}$.

Plugging this into the second equation, we have

$$2(3)+y \equiv 4 \pmod{7},$$

so the system has the solution $x \equiv 3 \pmod{7}$, $y \equiv 5 \pmod{7}$.

(d) We use the fact that $13 \equiv 1 \pmod{12}$. Thus, we can rewrite the equation as

$$x \equiv 13^{2023} \equiv 1^{2023} \equiv 1 \pmod{12}.$$

(e) One way to solve exponentiation problems is to test values until one identifies a pattern.

$$\begin{aligned}
&7^1 \equiv 7 \pmod{11} \\
&7^2 \equiv 49 \equiv 5 \pmod{11} \\
&7^3 = 7 \cdot 7^2 \equiv 7 \cdot 5 \equiv 2 \pmod{11} \\
&7^4 = 7 \cdot 7^3 \equiv 7 \cdot 2 \equiv 3 \pmod{11} \\
&7^5 = 7 \cdot 7^4 \equiv 7 \cdot 3 \equiv 10 \equiv -1 \pmod{11}
\end{aligned}$$

We theoretically could continue this until we the sequence starts repeating. However, notice that if $7^5 \equiv -1 \implies 7^{10} = (7^5)^2 \equiv (-1)^2 \equiv 1 \pmod{11}$.

Similarly, $7^{60} = (7^{10})^6 \equiv 1^6 \equiv 1 \pmod{11}$. As a final step, we have $7^{62} = 7^2 \cdot 7^{60} \equiv 7^2 \cdot 1 = 49 \equiv 5 \pmod{11}$.

2 Short Answer: Modular Arithmetic

Note 6

For each subpart, show your work and justify your answers.

(a) What is the multiplicative inverse of $n-1$ modulo n? (Your answer should be an expression that may involve n)

(b) What is the solution to the equation $3x \equiv 6 \pmod{17}$?

(c) Let $R_0 = 0; R_1 = 2; R_n = 4R_{n-1} - 3R_{n-2}$ for $n \geq 2$. Is $R_n \equiv 2 \pmod{3}$ for $n \geq 1$? (True or False)

(d) Given that $(7)(53) - m = 1$, what is the solution to $53x + 3 \equiv 10 \pmod{m}$? (Answer should be an expression that is interpreted $\pmod{m}$, and shouldn't consist of fractions.)

Solution:

(a) The answer is $n-1 \pmod{n}$. We can see this by noting that it is $-1 \pmod{n}$, or more directly, $(n-1)(n-1) \equiv n^2 - 2n + 1 \equiv 1 \pmod{n}$.

(b) The answer is $x \equiv 2 \pmod{17}$. Muliply both sides by 6 (the multiplicative inverse of 3 modulo 17) and reduce.

(c) The statement is true. We can see this by taking the recursive formula modulo 3. This gives us that $R_n \equiv R_{n-1} \pmod{3}$, hence since $R_1 \equiv 2 \pmod{3}$, every R_i must also be 2 modulo 3.

(d) Note that since $7 \cdot 53 - m = 1$, we can take both sides modulo m and find that $7 \cdot 53 \equiv 1 \pmod{m}$, hence 7 is the inverse of 53 modulo m. Thus, we can solve the equation by subtracting by 3 on both sides and multiplying by 7, giving that $x \equiv 49 \pmod{m}$.

3 Wilson's Theorem

Note 6 Wilson's Theorem states the following is true if and only if p is prime:

$$(p-1)! \equiv -1 \pmod{p}.$$

Prove both directions (it holds if AND only if p is prime).

Hint for the if direction: Consider rearranging the terms in $(p-1)! = 1 \cdot 2 \cdot \cdots \cdot (p-1)$ to pair up terms with their inverses, when possible. What terms are left unpaired?

Hint for the only if direction: If p is composite, then it has some prime factor q. What can we say about $(p-1)! \pmod{q}$?

Solution:

Direction 1: If p is prime, then the statement holds.

For the integers $1, \cdots, p-1$, every number has an inverse. However, it is not possible to pair a number off with its inverse when it is its own inverse. This happens when $x^2 \equiv 1 \pmod{p}$, or when $p \mid x^2 - 1 = (x-1)(x+1)$. Thus, $p \mid x-1$ or $p \mid x+1$, so $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$. Thus, the only integers from 1 to $p-1$ inclusive whose inverse is the same as itself are 1 and $p-1$.

We reconsider the product $(p-1)! = 1 \cdot 2 \cdots p-1$. The product consists of 1, $p-1$, and pairs of numbers with their inverse, of which there are $\frac{p-1-2}{2} = \frac{p-3}{2}$. The product of the pairs is 1 (since the product of a number with its inverse is 1), so the product $(p-1)! \equiv 1 \cdot (p-1) \cdot 1 \equiv -1 \pmod{p}$, as desired.

Direction 2: The expression holds *only if* p is prime (contrapositive: if p isn't prime, then it doesn't hold).

We will prove by contradiction that if some number p is composite, then $(p-1)! \not\equiv -1 \pmod{p}$. Suppose for contradiction that $(p-1)! \equiv -1 \pmod{p}$. Note that this means we can write $(p-1)!$ as $p \cdot k - 1$ for some integer k.

Since p isn't prime, it has some prime factor q where $2 \le q \le n-2$, and we can write $p = q \cdot r$. Plug this into the expression for $(p-1)!$ above, yielding us $(p-1)! = (q \cdot r)k - 1 = q(rk) - 1 \implies (p-1)! \equiv -1 \pmod{q}$. However, we know q is a term in $(p-1)!$, so $(p-1)! \equiv 0 \pmod{q}$. Since $0 \not\equiv -1 \pmod{q}$, we have reached our contradiction.

4 Celebrate and Remember Textiles

Note 6 You've decided to knit a 70-themed baby blanket as a gift for your cousin and want to incorporate rows from three different stitch patterns with the following requirements on the row lengths of each of the stitch patterns:

- Alternating Link: Multiple of 7, plus 4
- Double Broken Rib: Multiple of 4, plus 2
- Swag: Multiple of 5, plus 2

You want to be able to switch between knitting these different patterns without changing the number of stitches on the needle, so you must use a number of stitches that simultaneously meets the requirements of all three patterns.

Find the *smallest number of stitches* you need to cast on in order to incorporate all three patterns in your baby blanket.

Solution: Let x be the number of stitches we need to cast on. Using the Chinese Remainder Theorem, we can write the following system of congruences:

$$\begin{aligned} x &\equiv 4 \pmod{7} \\ x &\equiv 2 \pmod{4} \\ x &\equiv 2 \pmod{5}. \end{aligned}$$

We have $M = 7 \cdot 4 \cdot 5 = 140$, $r_1 = 4$, $m_1 = 7$, $b_1 = M/m_1 = 4 \cdot 5 = 20$, $r_2 = 3$, $m_2 = 4$, $b_2 = M/m_2 = 7 \cdot 5 = 35$, and $r_3 = 2$, $m_3 = 5$, $b_3 = M/m_3 = 7 \cdot 4 = 28$. We need to solve for the multiplicative inverse of b_i modulo m_i for $i \in \{1,2,3\}$:

$$\begin{aligned} b_1 a_1 &\equiv 1 \pmod{m_1} \\ 20a_1 &\equiv 1 \pmod{7} \\ 6a_1 &\equiv 1 \pmod{7} \\ \rightarrow a_1 &= 6, \end{aligned}$$

$$\begin{aligned} b_2 a_2 &\equiv 1 \pmod{m_2} \\ 35a_2 &\equiv 1 \pmod{4} \\ 3a_2 &\equiv 1 \pmod{4} \\ \rightarrow a_2 &= 3, \end{aligned}$$

and

$$\begin{aligned} b_3 a_3 &\equiv 1 \pmod{m_3} \\ 28a_3 &\equiv 1 \pmod{5} \\ 3a_3 &\equiv 1 \pmod{5} \\ \rightarrow a_3 &= 2. \end{aligned}$$

Therefore,

$$\begin{aligned} x &\equiv 6 \cdot 20 \cdot 4 + 2 \cdot 35 \cdot 3 + 2 \cdot 28 \cdot 2 \pmod{140} \\ &\equiv 102 \pmod{140}, \end{aligned}$$

so the smallest x that satisfies all three congruences is 102. Therefore we should cast on 102 stitches in order to be able to knit all three patterns into the blanket.

5 Euler's Totient Theorem

Note 6
Note 7

Euler's Totient Theorem states that, if n and a are coprime,

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

where $\phi(n)$ (known as Euler's Totient Function) is the number of positive integers less than or equal to n which are coprime to n (including 1). Note that this theorem generalizes Fermat's Little Theorem, since if n is prime, then $\phi(n) = n-1$.

(a) Let the numbers less than n which are coprime to n be $m_1, m_2, \ldots, m_{\phi(n)}$. Argue that the set

$$\{am_1, am_2, \ldots, am_{\phi(n)}\}$$

is a permutation of the set

$$\{m_1, m_2, \ldots, m_{\phi(n)}\}.$$

In other words, prove that

$$f : \{m_1, m_2, \ldots, m_{\phi(n)}\} \to \{m_1, m_2, \ldots, m_{\phi(n)}\}$$

is a bijection, where $f(x) := ax \pmod{n}$.

(b) Prove Euler's Theorem. (Hint: Recall the FLT proof.)

Solution:

(a) This problem mirrors the proof of Fermat's Little Theorem, except now we work with the set $\{m_1, m_2, \cdots, m_{\phi(n)}\}$.

Since m_i and a are both coprime to n, so is $a \cdot m_i$. Suppose $a \cdot m_i$ shared a common factor with n, and WLOG, assume that it is a prime p. Then, either $p|a$ or $p|m_i$. In either case, p is a common factor between n and one of a or m_i, contradiction.

We now prove that f is injective. Suppose we have $f(x) = f(y)$, so $ax \equiv ay \pmod{n}$. Since a has a multiplicative inverse $\pmod{n}$, we see $x \equiv y \pmod{n}$, thus showing that f is injective.

We continue to show that f is surjective. Take any y that is relatively prime to n. Then, we see that $f(a^{-1}y) \equiv y \pmod{n}$, so therefore, there is an x such that $f(x) = y$. Furthermore, $a^{-1}y \pmod{n}$ is relatively prime to n, since we are multiplying two numbers that are relatively prime to n.

(b) Since both sets have the same elements, just in different orders, multiplying them together gives

$$m_1 \cdot m_2 \cdot \ldots \cdot m_{\phi(n)} \equiv am_1 \cdot am_2 \cdot \ldots \cdot am_{\phi(n)} \pmod{n}$$

and factoring out the a terms,

$$m_1 \cdot m_2 \cdot \ldots \cdot m_{\phi(n)} \equiv a^{\phi(n)} \left(m_1 \cdot m_2 \cdot \ldots \cdot m_{\phi(n)}\right) \pmod{n}.$$

Thus we have $a^{\phi(n)} \equiv 1 \pmod{n}$.

6 Sparsity of Primes

Note 6

A prime power is a number that can be written as p^i for some prime p and some positive integer i. So, $9 = 3^2$ is a prime power, and so is $8 = 2^3$. $42 = 2 \cdot 3 \cdot 7$ is not a prime power.

Prove that for any positive integer k, there exists k consecutive positive integers such that none of them are prime powers.

Hint: This is a Chinese Remainder Theorem problem. We want to find n such that $(n+1)$, $(n+2)$, $\ldots$, and $(n+k)$ are all not powers of primes. We can enforce this by saying that $n+1$ through $n+k$ each must have two distinct prime divisors. In your proof, you can choose these prime divisors arbitrarily.

Solution:

We want to find n such that $n+1, n+2, n+3, \ldots, n+k$ are all not powers of primes. We can enforce this by saying that $n+1$ through $n+k$ each must have two distinct prime divisors. So, select $2k$ primes, $p_1, p_2, \ldots, p_{2k}$, and enforce the constraints

$$\begin{aligned}
n+1 &\equiv 0 \pmod{p_1 p_2} \\
n+2 &\equiv 0 \pmod{p_3 p_4} \\
&\vdots \\
n+i &\equiv 0 \pmod{p_{2i-1} p_{2i}} \\
&\vdots \\
n+k &\equiv 0 \pmod{p_{2k-1} p_{2k}}.
\end{aligned}$$

By Chinese Remainder Theorem, we can calculate the value of n, so this n must exist, and thus, $n+1$ through $n+k$ are not prime powers.

What's even more interesting here is that we could select any $2k$ primes we want!