HW 07 (Optional)

1 Proofs of the Combinatorial Variety

Note 10

Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a,b,c) such that $a+b+c = m$.)

Solution:

(a) Suppose we have n people and want to pick some of them to form a special committee. Moreover, suppose we want to pick a leader from among the committee members - how many ways can we do this?

We can do so by first picking the committee members, and then choosing the leader from among the chosen members. We can pick a committee of size k in $\binom{n}{k}$ ways, and once we have picked the committee, we have k choices for which member becomes the leader. In order to account for all possible committee sizes, we need to sum over all valid values of k, hence we get the expression

$$\sum_{k=0}^{n} k \cdot \binom{n}{k},$$

which is exactly the left hand side of the identity we want to prove.

Now, we can also count this set by first picking the leader for the committee, then choosing the rest of committee. We have n choices for the leader, and then among the remaining $n-1$ people, we can pick any subset to form the rest of the committee. Picking a subset of size k can be done in $\binom{n-1}{k}$ ways, hence summing over k, we get the expression

$$n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k},$$

which is exactly the right hand side of the identity we want to prove.

(b) Suppose we have n distinguishable red pencils, n distinguishable blue pencils, and n distinguishable green pencils ($3n$ pencils total), and want to choose m of these pencils to bring to class. How many ways can be do this?

We can do so by just picking the m pencils without considering color, as they are all distinguishable. There are $\binom{3n}{m}$ ways of doing this, which is exactly the right hand side of the identity we want to prove.

We can also count this set by picking some red pencils, the picking some blue pencils, and then finally picking some green pencils. We can pick a red pencils, b blue pencils, and c green pencils (with the tacit assumption that $a+b+c=m$) in $\binom{n}{a}\cdot\binom{n}{b}\cdot\binom{n}{c}$ ways. Finally, in order to account for all possible distributions of pencils, we need to sum over all valid triples (a,b,c), which gives us the expression

$$\sum_{a+b+c=m}\binom{n}{a}\cdot\binom{n}{b}\cdot\binom{n}{c},$$

which is exactly the left hand side of the identity we want to prove.

2 Fibonacci Fashion

Note 10

You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets $S_1, S_2, \ldots, S_t$, where $S_1 \subseteq \{1,2,\ldots,n\}$ and for $2 \le i \le t$, $S_i \subseteq \{1,2,\ldots,n\}$ and S_i is disjoint from S_{i-1}.

(a) For $t \ge 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).

(b) Use a combinatorial proof to prove the following identity, which, for $t \ge 1$ and $n \ge 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no accessory is worn two days in a row:

$$\sum_{x_1\ge 0}\sum_{x_2\ge 0}\cdots\sum_{x_t\ge 0}\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)

Solution:

(a) We will prove this by strong induction.

Base cases: For $k = 1$, the only binary strings possible are 0 and 1. Therefore, there are two possible binary strings, and $F_{k+2} = F_3 = 2$. For $k = 2$, the binary strings possible are 11, 01, and 10, and we have $F_{k+2} = F_4 = 3$, so the identity holds.

Inductive hypothesis: For $k \geq 2$, assume that for all $1 \leq x \leq k$, there are F_{x+2} binary strings of length x with no consecutive zeros.

Inductive step: Consider the set of binary strings of length $k+1$ with no consecutive zeros. We can group these into two sets: those which end with 0, and those which end with 1.
For those that end with a 0, these can be constructed by taking the set of binary strings of length $k-1$ with no consecutive zeros and appending 10 to the end of them. Then by the inductive hypothesis, this set is of size F_{k+1}. For those that end with a 1, these can be constructed by taking the set of binary strings of length k with no consecutive zeros and appending a 1 to the end of them. Then by the inductive hypothesis, this set is of size F_{k+2}.
Since the union of these two subsets (those which end with 0 and those which end with 1) cover all possible elements in the set of binary strings of length $k+1$ with no consecutive zeros, the size of this set will be $F_{k+1} + F_{k+2} = F_{k+3}$. This thus proves the inductive hypothesis.

(b) We first consider the left-hand-side of the identity. To create subsets of accessories that are consecutively disjoint with sizes $x_i = |S_i|$, $1 \leq i \leq n$, there are $\binom{n}{x_1}$ ways to create S_1, the subset of accessories you will wear on the first day. Then since S_2 must be disjoint from S_1, there are $\binom{n-x_1}{x_2}$ ways choose accessories to create S_2. Since S_3 must be disjoint from S_2, there are $\binom{n-x_2}{x_3}$ ways choose accessories to create S_3, and so on. Thus there are $\binom{n}{x_1}\binom{n-x_1}{x_2}\cdots\binom{n-x_{t-1}}{x_t}$ ways to create subsets of accessories $S_1, \ldots, S_t$ with respective sizes $x_1, \ldots, x_t$. Then altogether, $S_1, \ldots, S_t$ can be created in

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_2}{x_3}\cdots\binom{n-x_{t-1}}{x_t}$$

ways.

Now, consider the right-hand-side of the identity. Now for each accessory $i \in \{1, \ldots, n\}$, we will first decide which subsets $S_1, \ldots, S_t$ will contain accessory i, where we can't assign item i to consecutive subsets. For each accessory, we create a binary string of length t, where the leading digit represents S_1, the next digit represents S_2, and so on. We will say that a 0 in digit k means that we will wear the accessory on day k. Therefore, the number of ways we can assign accessory i to subsets $S_1, \ldots, S_t$ such that no two consecutive subsets both have accessory i is the same as the number of binary strings of length t with no consecutive zeros. Thus using the result in part (a), there are F_{t+2} ways to select the nonconsecutive subsets containing i among $S_1, \ldots, S_t$. Since we have n accessories, accessories $1, \ldots, n$ can be placed into subsets $S_1, \ldots, S_t$ in $(F_{t+2})^n$ ways.

This thus proves the identity.

3 Unions and Intersections

Note 11 Given:

- X is a countable, non-empty set. For all $i \in X$, A_i is an uncountable set.
- Y is an uncountable set. For all $i \in Y$, B_i is a countable set.

For each of the following, decide if the expression is "Always countable", "Always uncountable", "Sometimes countable, Sometimes uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $X \cap Y$

(b) $X \cup Y$

(c) $\bigcup_{i \in X} A_i$

(d) $\bigcap_{i \in X} A_i$

(e) $\bigcup_{i \in Y} B_i$

(f) $\bigcap_{i \in Y} B_i$

Solution:

(a) Always countable. $X \cap Y$ is a subset of X, which is countable.

(b) Always uncountable. $X \cup Y$ is a superset of Y, which is uncountable.

(c) Always uncountable. Let x be any element of X. A_x is uncountable. Thus, $\bigcup_{i \in X} A_i$, a superset of A_x, is uncountable.

(d) Sometimes countable, sometimes uncountable.

Countable: When the A_i are disjoint, the intersection is empty, and thus countable. For example, let $X = \mathbb{N}$, let $A_i = \{i\} \times \mathbb{R} = \{(i,x) \mid x \in \mathbb{R}\}$. Then, $\bigcap_{i \in X} A_i = \varnothing$.

Uncountable: When the A_i are identical, the intersection is uncountable. Let $X = \mathbb{N}$, let $A_i = \mathbb{R}$ for all i. $\bigcap_{i \in X} A_i = \mathbb{R}$ is uncountable.

(e) Sometimes countable, sometimes uncountable.

Countable: Make all the B_i identical. For example, let $Y = \mathbb{R}$, and $B_i = \mathbb{N}$. Then, $\bigcup_{i \in Y} B_i = \mathbb{N}$ is countable.

Uncountable: Let $Y = \mathbb{R}$. Let $B_i = \{i\}$. Then, $\bigcup_{i \in Y} B_i = \mathbb{R}$ is uncountable.

(f) Always countable. Let y be any element of Y. B_y is countable. Thus, $\bigcap_{i \in Y} B_i$, a subset of B_y, is also countable.

4 Countability Proof Practice

Note 11

(a) A disk is a 2D region of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq r^2\}$, for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Say you have a set of disks in $\mathbb{R}^2$ such that none of the disks overlap (with possibly varying x_0, y_0, and r values). Is this set always countable, or potentially uncountable?

(*Hint*: Attempt to relate it to a set that we know is countable, such as $\mathbb{Q} \times \mathbb{Q}$.)

(b) A circle is a subset of the plane of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 = r^2\}$ for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Now say you have a set of circles in $\mathbb{R}^2$ such that none of the circles overlap (with possibly varying x_0, y_0, and r values). Is this set always countable, or potentially uncountable?

(*Hint*: The difference between a circle and a disk is that a disk contains all of the points in its interior, whereas a circle does not.)

Solution:

(a) Countable. Each disk must contain at least one rational point (an (x,y)-coordinate where $x, y \in \mathbb{Q}$) in its interior, and due to the fact that no two disks overlap, the cardinality of the set of disks can be no larger than the cardinality of $\mathbb{Q} \times \mathbb{Q}$, which we know to be countable.

(b) Possibly uncountable. Consider the circles $C_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r\}$ for each $r \in \mathbb{R}$. For $r_1 \neq r_2$, C_{r_1} and C_{r_2} do not overlap, and there are uncountably many of these circles (one for each real number).