CS 70 Discrete Mathematics and Probability Theory
Fall 2024 Rao, Hug HW 07 (Optional)

Due: -
Grace period until -

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Proofs of the Combinatorial Variety

Note 10 Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a,b,c) such that $a+b+c=m$.)

2 Fibonacci Fashion

Note 10 You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets $S_1, S_2, \dots, S_t$, where $S_1 \subseteq \{1,2,\dots,n\}$ and for $2 \le i \le t$, $S_i \subseteq \{1,2,\dots,n\}$ and S_i is disjoint from S_{i-1}.

(a) For $t \ge 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).

(b) Use a combinatorial proof to prove the following identity, which, for $t \geq 1$ and $n \geq 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no accessory is worn two days in a row:

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-x_2}{x_3} \cdots \binom{n-x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)

3 Unions and Intersections

Note 11 Given:

- X is a countable, non-empty set. For all $i \in X$, A_i is an uncountable set.
- Y is an uncountable set. For all $i \in Y$, B_i is a countable set.

For each of the following, decide if the expression is "Always countable", "Always uncountable", "Sometimes countable, Sometimes uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $X \cap Y$

(b) $X \cup Y$

(c) $\bigcup_{i \in X} A_i$

(d) $\bigcap_{i \in X} A_i$

(e) $\bigcup_{i \in Y} B_i$

(f) $\bigcap_{i \in Y} B_i$

4 Countability Proof Practice

Note 11 (a) A disk is a 2D region of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq r^2\}$, for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Say you have a set of disks in $\mathbb{R}^2$ such that none of the disks overlap (with possibly varying x_0, y_0, and r values). Is this set always countable, or potentially uncountable?

(*Hint*: Attempt to relate it to a set that we know is countable, such as $\mathbb{Q} \times \mathbb{Q}$.)

(b) A circle is a subset of the plane of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 = r^2\}$ for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. Now say you have a set of circles in $\mathbb{R}^2$ such that none of the circles overlap (with possibly varying x_0, y_0, and r values). Is this set always countable, or potentially uncountable?

(*Hint*: The difference between a circle and a disk is that a disk contains all of the points in its interior, whereas a circle does not.)