## 1 Probability Warm-Up

Note 13
(a) Suppose that we have a bucket of 30 green balls and 70 orange balls. If we pick 15 balls uniformly out of the bucket, what is the probability of getting exactly $k$ green balls (assuming $0 \leq k \leq 15$ ) if the sampling is done with replacement, i.e. after we take a ball out the bucket we return the ball back to the bucket for the next round?
(b) Same as part (a), but the sampling is without replacement, i.e. after we take a ball out the bucket we do not return the ball back to the bucket.
(c) If we roll a regular, 6 -sided die 5 times. What is the probability that at least one value is observed more than once?

## Solution:

(a) Let $A$ be the event of getting exactly $k$ green balls. Then treating all balls as distinguishable, we have a total of $100^{15}$ possibilities to draw a sequence of 15 balls. In order for this sequence to have exactly $k$ green balls, we need to first assign them one of $\binom{15}{k}$ possible locations within the sequence. Once done so, we have $30^{k}$ ways of actually choosing the green balls, and $70^{15-k}$ possibilities for choosing the orange balls. Thus in total we arrive at

$$
\mathbb{P}[A]=\frac{\binom{15}{k} \cdot 30^{k} \cdot 70^{15-k}}{100^{15}}=\binom{15}{k}\left(\frac{3}{10}\right)^{k}\left(\frac{7}{10}\right)^{15-k}
$$

(b) Using a similar approach, there are a total of $100 \cdot 99 \cdots 86=\frac{100!}{(100-15)!}=\frac{100!}{85!}$ ways to grab 15 balls.
We still want $k$ green balls and $15-k$ orange balls, which we can select in

$$
\begin{aligned}
& \binom{15}{k} \cdot(30 \cdot 29 \cdots(30-(k-1))) \cdot(70 \cdot 69 \cdots(70-(15-k-1))) \\
& =\binom{15}{k} \frac{30!}{(30-k)!} \cdot \frac{70!}{(70-(15-k))!}
\end{aligned}
$$

ways. Here, we have $\binom{15}{k}$ possible locations for the $k$ green balls, and we use permutations rather than combinations to account for the fact that we are picking balls without replacement.
Since our sample space is uniform, the probability is thus

$$
\mathbb{P}[A]=\frac{\binom{15}{k} \frac{30!}{(30-k)!} \cdot \frac{70!}{(70-(15-k))!}}{\frac{100!}{(100-15)!}}=\frac{\binom{15}{k} \frac{30!}{(30-k)!} \cdot \frac{70!}{(55+k)!}}{\frac{100!}{85!}}
$$

Alternative Solution: Instead of considering a probability space where each ball is distinct, we can also consider a probability space where each ball is indistinguishable, and order does not matter. Intuitively, this is equivalent to the previous solution, since the numerator and denominator are accounting for order in the exact same way; considering the same situation without order and with indistinguishable balls is equivalent to dividing both the numerator and denominator by 15 ! for the number of arrangements of the 15 balls we select.
With this in mind, we note that the size of the sample space is now $\binom{100}{15}$, since we are choosing 15 balls out of a total of 100 . To find $|A|$, we need to be able to find out how many ways we can choose $k$ green balls and $15-k$ orange balls. This means that we have $|A|=\binom{30}{k}\binom{70}{15-k}$, since we must select $k$ green balls out of 30 total, and $15-k$ orange balls out of 70 total.
Putting this together, we have

$$
\mathbb{P}[A]=\frac{\binom{30}{k}\binom{70}{15-k}}{\binom{100}{15}} .
$$

(c) Let $B$ be the event that at least one value is observed more than once. We see that $\mathbb{P}[B]=$ $1-\mathbb{P}[\bar{B}]$. So we need to find out the probability that the values of the 5 rolls are distinct. We see that $\mathbb{P}[\bar{B}]$ simply the number of ways to choose 5 numbers (order matters) divided by the sample space (which is $6^{5}$ ). So

$$
\mathbb{P}[\bar{B}]=\frac{6!}{6^{5}}=\frac{5!}{6^{4}} .
$$

And

$$
\mathbb{P}[B]=1-\frac{5!}{6^{4}}
$$

## 2 Five Up

Note 13
Say you toss a coin five times, and record the outcomes. For the three questions below, you can assume that order matters in the outcome, and that the probability of heads is some $p$ in $0<p<1$, but not that the coin is fair $(p=0.5)$.
(a) What is the size of the sample space, $|\Omega|$ ?
(b) How many elements of $\Omega$ have exactly three heads?
(c) How many elements of $\Omega$ have three or more heads?

For the next three questions, you can assume that the coin is fair (i.e. heads comes up with $p=0.5$, and tails otherwise).
(d) What is the probability that you will observe the sequence HHHTT? What about HHHHT?
(e) What is the probability of observing at least one head?
(f) What is the probability you will observe more heads than tails?

For the final three questions, you can instead assume the coin is biased so that it comes up heads with probability $p=\frac{2}{3}$.
(g) What is the probability of observing the outcome HHHTT? What about HHHHT?
(h) What about the probability of at least one head?
(i) What is the probability you will observe more heads than tails?

## Solution:

(a) Since for each coin toss, we can have either heads or tails, we have $2^{5}$ total possible outcomes.
(b) Since we know that we have exactly 3 heads, what distinguishes the outcomes is at which point these heads occurred. There are 5 possible places for the heads to occur, and we need to choose 3 of them, giving us the following result: $\binom{5}{3}$.
(c) We can use the same approach from part (b), but since we are asking for 3 or more, we need to consider the cases of exactly 4 heads, and exactly 5 heads as well. This gives us the result as: $\binom{5}{3}+\binom{5}{4}+\binom{5}{5}=16$.
To see why the number is exactly half of the total number of outcomes, denote the set of outcomes that has 3 or more heads as $A$. If we flip over every coin in each outcome in set $A$, we get all the outcomes that have 2 or fewer heads. Denote the new set $\bar{A}$. Then we know that $A$ and $\bar{A}$ have the same size and they together cover the whole sample space. Therefore, $|A|=|\bar{A}|$ and $|A|+|\bar{A}|=2^{5}$, which gives $|A|=2^{5} / 2$.
(d) Since each coin toss is an independent event, the probability of each of the coin tosses is $\frac{1}{2}$ making the probability of this outcome $\frac{1}{2^{5}}$. This holds for both cases since both heads and tails have the same probability.
(e) We will use the complementary event, which is the event of getting no heads. The probability of getting no heads is the probability of getting all tails. This event has a probability of $\frac{1}{2^{5}}$ by a similar argument to the previous part. Since we are asking for the probability of getting at least one heads, our final result is: $1-\frac{1}{2^{5}}$.
(f) To have more heads than tails is to claim that we flip at least 3 heads. Since each outcome in this probability space is equally likely, we can divide the number of outcomes where there are 3 or more heads by the total number of outcomes to give us: $\frac{\binom{5}{3}+\binom{5}{4}+\binom{5}{5}}{2^{5}}=\frac{1}{2}$

Alternatively, we see that for every sequence with more heads than tails we can create a corresponding sequence with more tails than heads by "flipping" the bits. For example, a sequence HTHHT which has more heads than tails corresponds to a flipped sequence THTTH which has more tails than heads. As a result, for every sequence with more heads there's a sequence with more tails. Thus, the probability of having a sequence with more heads is $1 / 2$.
(g) By using the same idea of independence we get for HHHTT:

$$
\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3}=\frac{2^{3}}{3^{5}}
$$

For HHHHT, we get:

$$
\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}=\frac{2^{4}}{3^{5}}
$$

(h) Similar to the unbiased case, we will first find the probability of the complement event, which is having no heads. The probability of this is $\frac{1}{3^{5}}$, which makes our final result $1-\frac{1}{3^{5}}$.
(i) In this case, since we are working in a nonuniform probability space (getting 4 heads and 3 heads don't have the same probability), we need to separately consider the events with different numbers of heads to find our result. Thus for each $3 \leq i \leq 5$, we need to count the total number of outcomes with exactly $i$ heads and multiply it by the probability of achieving any of those outcomes. This yields:

$$
\begin{aligned}
\mathbb{P}[\geq 3 \text { Heads }] & =\binom{5}{3}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{2}+\binom{5}{4}\left(\frac{2}{3}\right)^{4}\left(\frac{1}{3}\right)^{1}+\binom{5}{5}\left(\frac{2}{3}\right)^{5}\left(\frac{1}{3}\right)^{0} \\
& =\binom{5}{3}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{2}+\binom{5}{4}\left(\frac{2}{3}\right)^{4}\left(\frac{1}{3}\right)+\binom{5}{5}\left(\frac{2}{3}\right)^{5} .
\end{aligned}
$$

## 3 Aces

Note 13 Note 14

Consider a standard 52-card deck of cards:
(a) Find the probability of getting an ace or a red card, when drawing a single card.
(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.
(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.
(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.
(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.
(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

Solution:
(a) Inclusion-Exclusion Principle: $\frac{4}{52}+\frac{26}{52}-\frac{2}{52}=\frac{28}{52}=\frac{7}{13}$.
(b) Inclusion-Exclusion, but we exclude the intersection: $\frac{4}{52}+\frac{13}{52}-2 \cdot \frac{1}{52}=\frac{15}{52}$.
(c) Ace of diamonds is fixed, but the other 4 cards in the hand can be any other card: $\frac{\binom{51}{4}}{\binom{52}{5}}$.
(d) Account for the number of ways to draw 2 aces and the number of ways to draw 3 non-aces: $\frac{\binom{4}{2} \cdot\binom{48}{3}}{\binom{52}{5}}$.
(e) Complement to getting no aces: $\mathbb{P}[$ at least one ace $]=1-\mathbb{P}[$ zero aces $]=1-\frac{\binom{48}{5}}{\binom{52}{5}}$.
(f) Complement to getting no aces and no hearts: $\mathbb{P}$ [at least one ace OR at least one heart $]=1-$ $\mathbb{P}[$ zero aces AND zero hearts $]=1-\frac{\binom{36}{5}}{\binom{52}{5}}$. This is because $52-13-3=36$, where 13 is the number of hearts and 3 is the number of non-heart aces.

## 4 Independent Complements

Note 14 Let $\Omega$ be a sample space, and let $A, B \subseteq \Omega$ be two independent events.
(a) Prove or disprove: $\bar{A}$ and $\bar{B}$ must be independent.
(b) Prove or disprove: $A$ and $\bar{B}$ must be independent.
(c) Prove or disprove: $A$ and $\bar{A}$ must be independent.
(d) Prove or disprove: It is possible that $A=B$.

## Solution:

(a) True. $\bar{A}$ and $\bar{B}$ must be independent:

$$
\begin{aligned}
\mathbb{P}[\bar{A} \cap \bar{B}] & =\mathbb{P}[\overline{A \cup B}] \\
& =1-\mathbb{P}[A \cup B] \\
& =1-(\mathbb{P}[A]+\mathbb{P}[B]-\mathbb{P}[A \cap B]) \\
& =1-\mathbb{P}[A]-\mathbb{P}[B]+\mathbb{P}[A] \mathbb{P}[B] \\
& =(1-\mathbb{P}[A])(1-\mathbb{P}[B])
\end{aligned}
$$

$$
=\mathbb{P}[\bar{A}] \mathbb{P}[\bar{B}] \quad(\text { since } \mathbb{P}[\bar{E}]=1-\mathbb{P}[E] \text { for all } E)
$$

(b) True. $A$ and $\bar{B}$ must be independent:

$$
\begin{aligned}
\mathbb{P}[A \cap \bar{B}] & =\mathbb{P}[A-(A \cap B)] \\
& =\mathbb{P}[A]-\mathbb{P}[A \cap B] \\
& =\mathbb{P}[A]-\mathbb{P}[A] \mathbb{P}[B] \\
& =\mathbb{P}[A](1-\mathbb{P}[B]) \\
& =\mathbb{P}[A] \mathbb{P}[\bar{B}]
\end{aligned}
$$

(c) False in general. If $0<\mathbb{P}[A]<1$, then $\mathbb{P}[A \cap A]=\mathbb{P}[\varnothing]=0$ but $\mathbb{P}[A] \mathbb{P}[A]>0$, so $\mathbb{P}[A \cap A] \neq$ $\mathbb{P}[A] \mathbb{P}[\bar{A}]$; therefore $A$ and $\bar{A}$ are not independent in this case.
(d) True. To give one example, if $\mathbb{P}[A]=\mathbb{P}[B]=0$, then $\mathbb{P}[A \cap B]=0=0 \times 0=\mathbb{P}[A] \mathbb{P}[B]$, so $A$ and $B$ are independent in this case. (Another example: If $A=B$ and $\mathbb{P}[A]=1$, then $A$ and $B$ are independent.)

## 5 Faulty Lightbulbs

Box 1 contains 1000 lightbulbs of which $10 \%$ are defective. Box 2 contains 2000 lightbulbs of which $5 \%$ are defective.
(a) Suppose a box is given to you at random and you randomly select a lightbulb from the box. If that lightbulb is defective, what is the probability you chose Box 1 ?
(b) Suppose now that a box is given to you at random and you randomly select two light- bulbs from the box. If both lightbulbs are defective, what is the probability that you chose from Box 1 ?

## Solution:

(a) Let:

- $D$ denote the event that the lightbulb we selected is defective.
- $B_{i}$ denote the event that the lightbulb we selected is from Box $i$.

We wish to compute $\mathbb{P}\left[B_{1} \mid D\right]$. Using Bayes' Rule we get:

$$
\begin{aligned}
\mathbb{P}\left[B_{1} \mid D\right] & =\frac{\mathbb{P}\left[D \mid B_{1}\right] \cdot \mathbb{P}\left[B_{1}\right]}{\mathbb{P}\left[B_{1}\right] \cdot \mathbb{P}\left[D \mid B_{1}\right]+\mathbb{P}\left[B_{2}\right] \cdot \mathbb{P}\left[D \mid B_{2}\right]} \\
& =\frac{0.1 \cdot 0.5}{0.5 \cdot 0.1+0.5 \cdot 0.05} \\
& =\frac{2}{3}
\end{aligned}
$$

(b) Let:

- $D^{\prime}$ denote the event that both the lightbulbs we selected are defective.
- $B_{i}$ denote the event that the lightbulb we selected is from Box $i$.

We wish to compute $\mathbb{P}\left[B_{1} \mid D^{\prime}\right]$. Using Bayes' Rule we get:

$$
\begin{aligned}
\mathbb{P}\left[B_{1} \mid D^{\prime}\right] & =\frac{\mathbb{P}\left[D^{\prime} \mid B_{1}\right] \cdot \mathbb{P}\left[B_{1}\right]}{\mathbb{P}\left[B_{1}\right] \cdot \mathbb{P}\left[D^{\prime} \mid B_{1}\right]+\mathbb{P}\left[B_{2}\right] \cdot \mathbb{P}\left[D^{\prime} \mid B_{2}\right]} \\
& =\frac{\frac{100}{1000} \cdot \frac{99}{999} \cdot 0.5}{0.5 \cdot \frac{100}{1000} \cdot \frac{99}{999}+0.5 \cdot \frac{100}{2000} \cdot \frac{99}{1999}} \\
& \approx 0.8
\end{aligned}
$$

