CS 70 Discrete Mathematics and Probability Theory
Fall 2026 Hug, Sabin

HW 09

Due: Wednesday, 11/4, 6:00 PM
Grace period until Friday, 11/6, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Conditional Probability and Combinations of Events Basics

Note 17
Note 18

Professor Hug rolls a fair 3-sided die numbered (1, 2, 3), and then Professor Sabin rolls a fair 4-sided die numbered (1, 2, 3, 4). Let A be the event that Professor Hug rolls an even number and B be the event that the sum of the professors' rolls is greater than or equal to 5.

(a) Let's first explore conditional probability, Bayes' Rule, and Total Probability.

(i) How many total outcomes are there? Are they equally likely?

(ii) Which outcomes are in event A? What is $\mathbb{P}[A]$?

(iii) Which outcomes are in event B? What is $\mathbb{P}[B]$?

(iv) Which outcomes are in event $A \cap B$? What is $\mathbb{P}[A \cap B]$?

(v) Use your answers above to calculate $\mathbb{P}[A \mid B]$.

(vi) Use Bayes' Rule to calculate $\mathbb{P}[B \mid A]$.

(vii) Which outcomes are in event $\overline{A} \cap B$? What is $\mathbb{P}[\overline{A} \cap B]$?

(viii) Finally, use the Law of Total Probability to calculate $\mathbb{P}[B]$. Does this agree with your answer to part (c)?

(b) Now, let's look at some combinations of events and independence.

(i) Which outcomes are in $A \cup B$? Calculate $\mathbb{P}[A \cup B]$.

(ii) Use the Inclusion-Exclusion Principle to calculate $\mathbb{P}[A \cup B]$. Does your answer agree with the previous part?

(iii) Are A and B mutually exclusive? Are they independent? Justify your answers.

2 Ghostly Conditions on Your Probability

Note 17

It's Halloween! You are in a class of 30 students, and start to observe everyone's costumes. Throughout, a random student is chosen uniformly at random from the class.

(a) You observe 12 people with ghost costumes and 18 students sharing candy. Of the students sharing candy, 8 of them are ghosts. What is the probability of a random student being a ghost, given that they are sharing candy?

(b) For the class in part (a), what is the probability of a random student sharing candy, being a ghost, or both?

(c) Each person can only wear 1 costume. You define the events: G = "the student is wearing a ghost costume", V = "the student is wearing a vampire costume", and C = "the student is sharing candy". Are the events G and V mutually exclusive? Are G and C mutually exclusive?

(d) You find that vampires just seem to be nicer than the other costumes. There are 10 students with vampire costumes. Any student with a vampire costume has a 90% chance of being one of the students sharing candy, while any student without a vampire costume has only a 45% chance of being a student who is sharing candy. If a student is selected uniformly at random, what is the probability that this student is sharing candy?

(e) For part (d), if you observe that a random student is sharing candy, what is the probability that they are wearing a vampire costume?

3 Box of Marbles

Note 17

You are given two boxes: one of them containing 900 red marbles and 100 blue marbles, the other one contains 500 red marbles and 500 blue marbles.

(a) If we pick one of the boxes randomly and pick a marble, what is the probability that it is blue?

(b) If we see that the marble is blue, what is the probability that it is chosen from box 1?

(c) Suppose we pick one marble from box 1 and without looking at its color we put it aside. Then we pick another marble from box 1. What is the probability that the second marble is blue?

4 Independent Complements

Note 18

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: $\overline{A}$ and $\overline{B}$ must be independent.

(b) Prove or disprove: A and $\overline{B}$ must be independent.

(c) Prove or disprove: A and $\overline{A}$ must be independent.

(d) Prove or disprove: It is possible that $A = B$.

5 Balls and Bins

Note 17
Note 18

Suppose you throw b balls into n labeled bins one at a time, where $b \geq 1$ and $n \geq 2$.

(a) What is the probability that the first bin is empty?

(b) What is the probability that the first k bins are empty?

(c) If k bins were randomly selected, what is the probability that they are all empty?

(d) Let A be the event that at least k bins are empty. Let m be the number of subsets of k bins out of the total n bins. If we assume A_i is the event that the ith subset of k bins is empty. Then we can write A as the union of A_i's:

$$A = \bigcup_{i=1}^{m} A_i.$$

Compute m in terms of n and k, and use the union bound to give an upper bound on $\mathbb{P}[A]$.

(e) What is the probability that the second bin is empty given that the first one is empty?

(f) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

(g) Are the events that "the first bin is empty" and "the second bin is empty" independent?

6 Last Dining Hall Visit

This is a stretch problem. Your solution does not need to be complete, but make a good-faith effort, and write your solution up to the point where you get stuck.

Note 17

Sunny has just enough flex dollars left to make one visit to the dining hall in the next n days before the semester ends. The menu changes every day, and she wants to make the most of her last visit.

She decides to use the following procedure to decide when to make the visit:

- For the next k days, she will check the day's menu.
- Starting from day $k+1$, she will visit the dining hall on the first day that she prefers the day's menu over the best menu she had observed in the first k days.

Assume she keeps a strict ranking of the menus offered at the dining hall for the n days, and that the menus appear in a uniformly random order.

After the semester is over, what is the probability that she visited the dining hall on the day that her favorite menu was served? (You may leave your answer as a summation.)

7 Socks

This is a stretch problem. Your solution does not need to be complete, but make a good-faith effort, and write your solution up to the point where you get stuck.

Note 17
Note 18

Suppose you have n different pairs of socks (n left socks and n right socks, for $2n$ individual socks total) in your dresser. You take the socks out of the dresser one by one without looking and lay them out in a row on the floor. In this question, we'll go through the computation of the probability that no two matching socks are next to each other.

(a) We can consider the sample space as the set of length $2n$ permutations. What is the size of the sample space Ω, and what is the probability of a particular permutation $\omega \in \Omega$?

(b) Let A_i be the event that the ith pair of matching socks are next to each other. Calculate $\mathbb{P}[A_i]$.

(c) Calculate $\mathbb{P}[A_1 \cap \ldots \cap A_k]$ for an arbitrary $k \geq 2$. (Hint: try using a counting based approach.)

(d) Putting these all together, calculate the probability that there is at least one pair of matching socks next to each other. Your answer can (and should) be expressed as a summation. (Hint: use Inclusion/Exclusion.)

(e) Using your answer from the previous part, what is the probability that no two matching socks are next to each other? (This should follow directly from your answer to the previous part, and also can be left as a summation.)