CS 70 Spring 2026 — Discrete Mathematics and Probability Theory — Sinclair, Song — HW 11

Due: Saturday, 4/11, 4:00 PM
Grace period until Saturday, 4/11, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Balls and Bins, All Day Every Day

Note 14
Note 15

Suppose n balls are thrown into n labeled bins one at a time, where n is a positive *even* integer.

(a) What is the probability that exactly k balls land in the first bin, where k is an integer $0 \le k \le n$?

(b) What is the probability p that at least half of the balls land in the first bin? (You may leave your answer as a summation.)

(c) Using the union bound, give a simple upper bound, in terms of p, on the probability that some bin contains at least half of the balls.

(d) What is the probability, in terms of p, that at least half of the balls land in the first bin, or at least half of the balls land in the second bin?

(e) After you throw the balls into the bins, you walk over to the bin which contains the first ball you threw, and you randomly pick a ball from this bin. What is the probability that you pick up the first ball you threw? (Again, leave your answer as a summation.)

2 Combined Head Count

Note 16

Suppose you flip a fair coin twice.

(a) What is the sample space Ω generated from these flips?

(b) Define a random variable X to be the number of heads. What is the distribution of X?

(c) Define a random variable Y to be 1 if you get a heads followed by a tails and 0 otherwise. What is the distribution of Y?

(d) Compute the conditional probabilities $\mathbb{P}[Y = i \mid X = j]$ for all combinations of i and j.

(e) Define a third random variable $Z = X + Y$. Use the conditional probabilities you computed in part (d) to find the distribution of Z.

3 Testing Model Planes

Note 16

Amin is testing model airplanes. He starts with n model planes which each independently have probability p of flying successfully each time they are flown, where $0 < p < 1$. Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Amin flying all remaining model planes and throwing away any that crash.

Let X_i be the random variable representing how many model planes remain after i days. (Note that $X_0 = n$.) Justify your answers for each part.

(a) What is the distribution of X_1? That is, what is $\mathbb{P}[X_1 = k]$?

(b) What is the distribution of X_2? That is, what is $\mathbb{P}[X_2 = k]$? Recognize the distribution of X_2 as one of the famous ones and provide its name and parameters.

(c) Repeat the previous part for X_t for arbitrary $t \geq 1$.

(d) What is the probability that at least one model plane still remains (has not crashed yet) after t days? Do not have any summations in your answer.

(e) Considering only the first day of flights, is the event A_1 that the first and second model planes crash independent from the event B_1 that the second and third model planes crash? Recall that two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$. Prove your answer using this definition.

(f) Considering only the first day of flights, let A_2 be the event that the first model plane crashes *and* exactly two model planes crash in total. Let B_2 be the event that the second plane crashes on the first day. What must n be equal to in terms of p such that A_2 is independent from B_2? Prove your answer using the definition of independence stated in the previous part.

(g) Are the random variables X_i and X_j, where $i < j$, independent? Recall that two random variables X and Y are independent if $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1]\,\mathbb{P}[Y = k_2]$ for all k_1 and k_2. Prove your answer using this definition.

4 Max/Min Dice Rolls

Note 16

Yining rolls three fair six-sided dice.

(a) Let X denote the maximum of the three values rolled. What is the distribution of X? That is, what is $\mathbb{P}[X = x]$ for each possible value of x? Leave your final answer in terms of x.

Hint: Try to first compute $\mathbb{P}[X \leq x]$ for each possible value of x. To check your answer, you can solve this problem using counting and ensure you get the same probabilities.

(b) Let Y denote the minimum of the three values rolled. What is the distribution of Y?

5 How Many Marbles?

Note 16

Leanne has 6 marbles, 2 red, 2 blue, and 2 green. She picks three marbles uniformly at random without replacement. Let X denote the number of blue marbles she draws.

(a) What is $\mathbb{P}[X=0]$, $\mathbb{P}[X=1]$, and $\mathbb{P}[X=2]$?

(b) What do the probabilities you computed in part (a) add up to?

(c) Compute $\mathbb{E}[X]$, using the definition of expectation.

(d) Suppose we define indicators X_i, $1 \leq i \leq 3$, where X_i is the indicator variable that equals 1 if the ith marble is a blue marble and 0 otherwise. Compute $\mathbb{E}[X]$ using linearity of expectation.

(e) Are the X_i indicators independent? Does this affect your solution to part (d)?

6 Swaps and Cycles

Note 16

A permutation of n objects is a bijection from $(1,\ldots,n)$ to itself. For example, the permutation $\pi = (2,1,4,3)$ of 4 objects is the mapping $\pi(1) = 2$, $\pi(2) = 1$, $\pi(3) = 4$, and $\pi(4) = 3$. We'll say that a permutation $\pi = (\pi(1),\ldots,\pi(n))$ contains a *swap* if there exist $i, j \in \{1,\ldots,n\}$ so that $\pi(i) = j$ and $\pi(j) = i$, where $i \neq j$. The example above contains two swaps: $(1,2)$ and $(3,4)$.

(a) In terms of n, what is the expected number of swaps in a random permutation?

(b) In the same spirit as above, π contains a *k-cycle* if there exist $i_1,\ldots,i_k \in \{1,\ldots,n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_k) = i_1$. What is the expected number of k-cycles?