Due: Saturday, 4/25, 4:00 PM
Grace period until Saturday, 4/25, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Estimating π

Note 19

In this problem, we discuss one way that you could probabilistically estimate π. We'll use a square dartboard of side length 2, and a circular target drawn inscribed in the square dartboard with radius 1. A dart is then thrown uniformly at random in the square. Let p be the probability that the dart lands inside the circle.

(a) What is the value of p?

(b) Suppose we throw N darts uniformly at random in the square. Let $\hat{p}$ be the proportion of darts that land inside the circle. Create an unbiased estimator X for π using $\hat{p}$ (in other words, create a random variable X such that $\mathbb{E}[X] = \pi$).

(c) Using Chebyshev's inequality, find a value of N such that your estimate X is within ε of π with $1-\delta$ confidence. Your answer should be in terms of ε and δ. Note that since we are estimating π, your answer should not have π in it, nor use any bounds on the value of π.

2 Tightness of Inequalities

Note 17

(a) Give an example where Markov's inequality is tight; that is, show that given any fixed $k > 0$, there exists a discrete non-negative random variable X which is **not always zero** such that $\mathbb{P}[X \geq k] = \mathbb{E}[X]/k$.

(b) Give an example where Chebyshev's inequality is tight; that is, show that given any fixed $k \geq 1$, there exists a random variable X such that $\mathbb{P}[|X - \mathbb{E}[X]| \geq k\sigma] = 1/k^2$, where $\sigma^2 = \text{Var}(X)$.

3 Max of Uniforms

Note 20

Let $X_1, \ldots X_n$ be independent Uniform$(0,1)$ random variables (i.e., they are each uniformly distributed on $[0,1]$), and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of n.

(a) What is the cdf of X?

(b) What is the pdf of X?

(c) What is $\mathbb{E}[X]$?

(d) What is $\mathrm{Var}(X)$?

4 Short Answer

Note 20

(a) Let X be uniform on the interval $[0,2]$, and define $Y = 4X^2 + 1$. Find the PDF, CDF, expectation, and variance of Y.

(b) Let X and Y have joint distribution

$$f(x,y) = \begin{cases} cxy + \frac{1}{4} & x \in [1,2] \text{ and } y \in [0,2] \\ 0 & \text{otherwise.} \end{cases}$$

Find the constant c (Hint: remember that the PDF must integrate to 1). Are X and Y independent?

(c) Let $X \sim \mathrm{Exp}(3)$.

(i) Find the probability that $X \in [0, 1]$.

(ii) Let $Y = \lfloor X \rfloor$, where the floor operator is defined as: $(\forall x \in [k, k+1))(\lfloor x \rfloor = k)$. For each $k \in \mathbb{N}$, what is the probability that $Y = k$? Write the distribution of Y in terms of one of the famous distributions; provide that distribution's name and parameters.

(d) Let $X_i \sim \mathrm{Exp}(\lambda_i)$ for $i = 1, \ldots, n$ be mutually independent. It is a (very nice) fact that $\min(X_1, \ldots, X_n) \sim \mathrm{Exp}(\mu)$. Find μ.

5 Darts with Friends

Note 20

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius 1 around the center. Alex's aim follows a uniform distribution over a disk of radius 2 around the center (independent of Michelle's distribution).

(a) Let the distance of Michelle's throw from the center be denoted by the random variable X and let the distance of Alex's throw from the center be denoted by the random variable Y.

(i) What's the cumulative distribution function of X?

(ii) What's the cumulative distribution function of Y?

(iii) What's the probability density function of X?

(iv) What's the probability density function of Y?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \max(X, Y)$?