CS 70	Discrete Mathematics and Probability Theory	
Spring 2026	Sinclair, Song	HW 14

Due: Monday 5/4, 4:00 PM
Grace period until Monday 5/4, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Predictable Gaussians

Note 20

Let Y be the result of a fair coin flip, and X be a normally distributed random variable with parameters dependent on Y. That is, if $Y = 1$, then $X \sim N(\mu_1, \sigma_1^2)$, and if $Y = 0$, then $X \sim N(\mu_0, \sigma_0^2)$.

(a) Sketch the two distributions of X overlaid on the same graph for the following cases:

(i) $\sigma_0^2 = \sigma_1^2, \mu_0 \neq \mu_1$

(ii) $\sigma_0^2 \neq \sigma_1^2, \mu_0 = \mu_1$

(b) Bayes' rule for mixed distributions can be formulated as $\mathbb{P}[Y = 1 \mid X = x] = \frac{\mathbb{P}[Y=1] f_{X|Y=1}(x)}{f_X(x)}$ where Y is a discrete distribution and X is a continuous distribution. Compute $\mathbb{P}[Y = 1 \mid X = x]$, and show that this can be expressed in the form of $\frac{1}{1+e^{\gamma}}$ for some expression γ. (Hint: any value z can be equivalently expressed as $e^{\ln(z)}$)

(c) In the special case where $\sigma_0^2 = \sigma_1^2$ find a simple expression for the value of x where $\mathbb{P}[Y = 1 \mid X = x] = \mathbb{P}[Y = 0 \mid X = x] = 1/2$, and interpret what the expression represents. (Hint: the identity $(a+b)(a-b) = a^2 - b^2$ may be useful)

2 Chebyshev's Inequality vs. Central Limit Theorem

Note 19
Note 20

Let n be a positive integer. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with the following distribution:

$$\mathbb{P}[X_i = -1] = \frac{1}{12}; \qquad \mathbb{P}[X_i = 1] = \frac{9}{12}; \qquad \mathbb{P}[X_i = 2] = \frac{2}{12}.$$

(a) Calculate the expectations and variances of X_1, $\sum_{i=1}^n X_i$, $\sum_{i=1}^n (X_i - \mathbb{E}[X_i])$, and

$$Z_n = \frac{\sum_{i=1}^n (X_i - \mathbb{E}[X_i])}{\sqrt{n/2}}.$$

(b) Use Chebyshev's Inequality to find an upper bound b for $\mathbb{P}[|Z_n| \geq 2]$.

(c) Use b from the previous part to bound $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$.

(d) As $n \to \infty$, what is the distribution of Z_n?

(e) We know that if $Z \sim \mathcal{N}(0,1)$, then $\mathbb{P}[|Z| \leq 2] = \Phi(2) - \Phi(-2) \approx 0.9545$. As $n \to \infty$, provide approximations for $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$.

3 Law of Large Numbers

Recall that the *Law of Large Numbers* holds if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{1}{n}S_n - \mathbb{E}\left[\frac{1}{n}S_n\right]\right| > \varepsilon\right] = 0.$$

In class, we saw that the Law of Large Numbers holds for $S_n = X_1 + \cdots + X_n$, where the X_i's are i.i.d. random variables. This problem explores if the Law of Large Numbers holds under other circumstances.

Packets are sent from a source to a destination node over the Internet. Each packet is sent on a certain route, and the routes are disjoint. Each route has a failure probability of $p \in (0,1)$ and different routes fail independently. If a route fails, all packets sent along that route are lost. You can assume that the routing protocol has no knowledge of which route fails.

For each of the following routing protocols, determine whether the Law of Large Numbers holds when S_n is defined as the total number of received packets out of n packets sent. Answer **Yes** if the Law of Large Number holds, or **No** if not. Give a justification of your answer. (Whenever convenient, you can assume that n is even.)

(a) **Yes** or **No**: Each packet is sent on a completely different route.

(b) **Yes** or **No**: The packets are split into $n/2$ pairs of packets. Each pair is sent together on its own route (i.e., different pairs are sent on different routes).

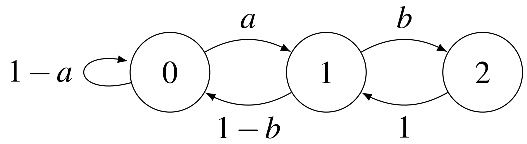
(c) **Yes** or **No**: The packets are split into 2 groups of $n/2$ packets. All the packets in each group are sent on the same route, and the two groups are sent on different routes.

(d) **Yes** or **No**: All the packets are sent on one route.

4 Analyze a Markov Chain

Note 21

Consider a Markov chain with the state diagram shown below where $a, b \in (0,1)$.



Here, we let $X(n)$ denote the state at time n.

(a) Is this Markov chain irreducible? Is this Markov chain aperiodic? Justify your answers.

(b) Calculate $\mathbb{P}[X(1)=1, X(2)=0, X(3)=1, X(4)=2 \mid X(0)=0]$.

(c) Calculate the invariant distribution. Do all initial distributions converge to this invariant distribution? Justify your answer.

5 A Bit of Everything

Note 21

Suppose that $X_0, X_1, \ldots$ is a Markov chain with finite state space $S = \{1, 2, \ldots, n\}$, where $n > 2$, and transition matrix P. Suppose further that

$$P(1,i) = \frac{1}{n} \quad \text{for all states } i \text{ and}$$
$$P(j, j-1) = 1 \quad \text{for all states } j \neq 1,$$

with $P(i,j) = 0$ everywhere else.

(a) Prove that this Markov chain is irreducible and aperiodic.

(b) Suppose you start at state 1. What is the distribution of T, where T is the number of transitions until you leave state 1 for the first time?

(c) Again starting from state 1, what is the expected number of transitions until you reach state n for the first time?

(d) Again starting from state 1, what is the probability you reach state 2 before you reach state n?

(e) Compute the stationary distribution of this Markov chain.