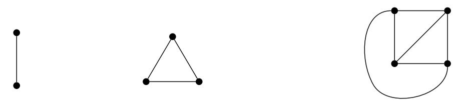
More on Graphs

Types of graphs.

- Complete Graphs.
- Trees.
- Planar Graphs.
- Hypercubes.

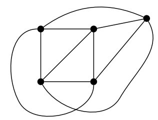
Complete Graph.



K_n complete graph on n vertices.
All edges are present.
Each vertex is adjacent to every other vertex.

How many edges?
Each vertex is incident to $n-1$ edges.
Sum of degrees is $n(n-1)$.
$\implies$ Number of edges is $n(n-1)/2$.
Remember sum of degree is $2|E|$.

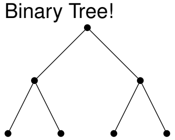
K_4 and K_5



K_5 is not planar.
Cannot be drawn in the plane without an edge crossing!
We will prove this later!

Trees!

Graph $G = (V,E)$.
Binary Tree!



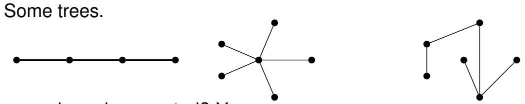
More generally.

Trees: Definitions

Definitions:

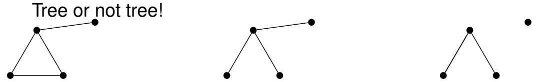
A connected graph without a cycle.
A connected graph with $|V|-1$ edges.
A connected graph where any edge removal disconnects it.
A connected graph where any edge addition creates a cycle.

Some trees.



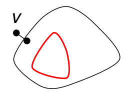
no cycle and connected? Yes.
$|V|-1$ edges and connected? Yes.
removing any edge disconnects it. Harder to check. but yes.
Adding any edge creates cycle. Harder to check. but yes.

Tree or not tree!



Equivalence of Definitions

Thm:
"G connected and has $|V|-1$ edges" $\equiv$
"G is connected and has no cycles."



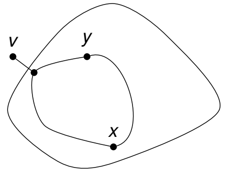
Proof of $\implies$ (only if): By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V|-1$ edges and has no cycles.

Induction Step: Assume for G with up to k vertices. Prove for $k+1$
Consider some vertex v in G. How is it connected to the rest of G?
Might it be connected by just 1 edge?
Is there a Degree 1 vertex?
Is the rest of G connected?

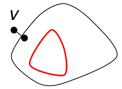
Equivalence of Definitions: Useful Lemma

Theorem:
"G connected and has $|V|-1$ edges" $\equiv$
"G is connected and has no cycles."

Lemma: If v is a degree 1 in connected graph G, $G-v$ is connected.
Proof:
For $x \neq v, y \neq v \in V$,
there is path between x and y in G since connected.
and does not use v (degree 1)
$\implies$ $G-v$ is connected. □



Proof of only if.



Thm:
"G connected and has $|V|-1$ edges" $\equiv$
"G is connected and has no cycles."

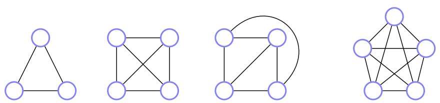
Proof of $\implies$: By induction on $|V|$.
Base Case: $|V|=1$. $0=|V|-1$ edges and has no cycles.
Induction Step: Assume for G with up to k vertices. Prove for $k+1$
Claim: There is a degree 1 node.
Proof: First, connected $\implies$ every vertex degree ≥ 1.
Sum of degrees is $2|V|-2$
Average degree $2-(2/|V|)$
Not everyone is bigger than average! □
By degree 1 removal lemma, $G-v$ is connected.
$G-v$ has $|V|-1$ vertices and $|V|-2$ edges so by induction
$\implies$ no cycle in $G-v$.
And no cycle in G since degree 1 cannot participate in cycle. □

Proof of "if part"

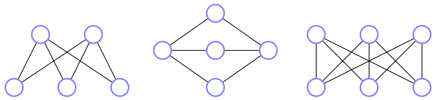
Thm:
"G is connected and has no cycles" $\implies$ "G connected and has $|V|-1$ edges"
Proof: Can we use the "degree 1" idea again?
Walk from a vertex using untraversed edges and vertices.
Until get stuck. Why? Finitely-many vertices, no cycle!
Claim: Degree 1 vertex.
Proof of Claim:
Can't visit more than once since no cycle.
Entered. Didn't leave. Only one incident edge. □
Removing node doesn't create cycle.
New graph is connected. (from our Degree 1 lemma).
By induction $G-v$ has $|V|-2$ edges.
G has one more or $|V|-1$ edges. □

Planar graphs.

A graph that can be drawn in the plane without edge crossings.

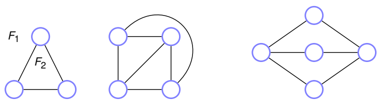


Planar? Yes for Triangle.
Four node complete K_4? Yes.
Five node complete or K_5 ? No! Why? Later.



Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or $K_{3,3}$. No. Why? Later.

Euler's Formula.



Faces: connected regions of the plane.

How many faces for
triangle? 2
complete on four vertices or K_4? 4
bipartite, complete two/three or $K_{2,3}$? 3

v is number of vertices, e is number of edges, f is number of faces.

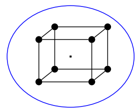
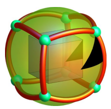
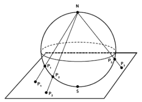
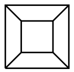
Euler's Formula: Connected planar graph has $v+f=e+2$.

Triangle: $3+2=3+2$!
K_4: $4+4=6+2$!
$K_{2,3}$: $5+3=6+2$!

Examples = 3! Proven! Not!!!!

Euler and Polyhedron.

Ancient Greek mathematicians knew formula for polyhedron.

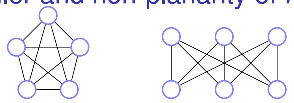
Faces? 6. Edges? 12. Vertices? 8.
Euler: Connected planar graph: $v+f=e+2$.
$8+6=12+2$.

Greeks couldn't prove it. Induction? Remove vertex for polyhedron?
Polyhedron without holes $\equiv$ Planar graphs.

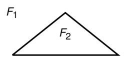
For Convex Polyhedron:
Surround by sphere.
Project from internal point polytope to sphere: drawing on sphere.
Project Sphere-N onto Plane: drawing on plane.

Euler proved formula thousands of years later!

Euler and non-planarity of K_5 and $K_{3,3}$



Euler: $v+f=e+2$ for connected planar graph.
We consider simple graphs where $v \geq 3$.
Consider Face edge Adjacencies

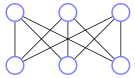


Each face is adjacent to at least three edges($v>2$).
$\geq 3f$ face-edge adjacencies.
Each edge is adjacent to two faces.
$=2e$ face-edge adjacencies.
$\implies 3f \leq 2e$ for any planar graph with $v>2$. Or $f \leq \frac{2}{3}e$.

Plug into Euler: $v+\frac{2}{3}e \geq e+2 \implies e \leq 3v-6$

K_5 Edges? $e=4+3+2+1=10$. Vertices? $v=5$.
$10 \not\leq 3(5)-6=9$. $\implies$ K_5 is not planar.

Proving non-planarity for $K_{3,3}$



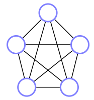
$K_{3,3}$? Edges = 9. Vertices = 6.

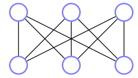
$e \leq 3(v)-6$ for planar graphs.

$9 \leq 3(6)-6$? Sure!

Need a different approach! See notes for details.

Summary: Planarity and Euler



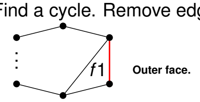


These graphs **cannot** be drawn in the plane without edge crossings.

Proof of Euler's formula.

Theorem (Euler): Connected planar graph has $v+f=e+2$.

Proof: Induction on e.
Base: $e=0$, $v=f=1$.
Induction Step:
First, if it is a tree: $e=v-1$, $f=1$, $v+1=(v-1)+2$. Done.
Suppose it is NOT a tree: Assume holds for $e \leq n$. Consider $e=n+1$.
Find a cycle. Remove edge.



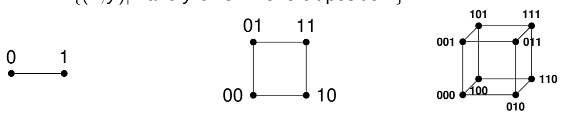
Joins two faces.
New graph: v-vertices. $e-1$ edges. $f-1$ faces. Planar.
$v+(f-1)=(e-1)+2$ by induction hypothesis.
Therefore $v+f=e+2$. □

Hypercubes.

Recall:
Complete graphs, really well connected! Lots of edges.
$|V|(|V|-1)/2$
Trees, connected, few edges.
$(|V|-1)$

Hypercubes. Well connected. $|V|\log|V|$ edges!
Also represents bit-strings nicely.

A hypercube is a graph $G=(V,E)$
$V=\{0,1\}^n$,
$E=\{(x,y)|x$ and y differ in one bit position.$\}$

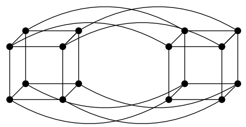


2^n vertices. number of n-bit strings!
$n2^{n-1}$ edges.
2^n vertices each of degree n
total degree is $n2^n$ and half as many edges!

Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An n-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x,1x)$.



Hypercube: Can't cut me!

Thm: Any subset S of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V-S$: $|E \cap S \times (V-S)| \geq |S|$

Terminology:
$(S, V-S)$ is cut.
$(E \cap S \times (V-S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Proof of Large Cuts.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side.
Proof:
Base Case: $n = 1$ V= $\{0,1\}$.
$S = \{0\}$ has one edge leaving.
$S = \emptyset$ has 0.

Induction Step Idea

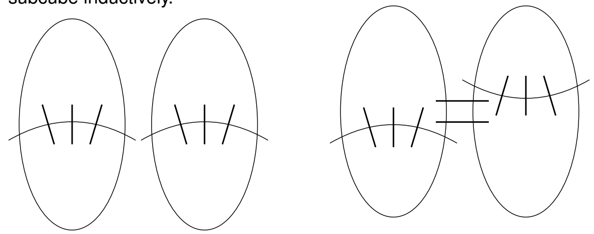
Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

Case 2: Count inside and across.



Induction Step

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.
Recursive definition:
$H_0 = (V_0, E_0), H_1 = (V_1, E_1)$, edges E_x that connect them.
$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
$S = S_0 \cup S_1$ where S_0 in first, and S_1 in other.

Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
Both S_0 and S_1 are small sides. So by induction.
Edges cut in $H_0 \geq |S_0|$.
Edges cut in $H_1 \geq |S_1|$.

Total cut edges $\geq |S_0| + |S_1| = |S|$. □

Induction Step. Case 2.

Thm: For any cut $(S, V-S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.
Proof: Induction Step. Case 2. $|S_0| \geq |V_0|/2$.
Recall Case 1: $|S_0|, |S_1| \leq |V|/2$
$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.
$\implies \geq |S_1|$ edges cut in E_1.
$|S_0| \geq |V_0|/2 \implies |V_0 - S_0| \leq |V_0|/2$
$\implies \geq |V_0| - |S_0|$ edges cut in E_0.

Edges in E_x connect corresponding nodes.
$\implies \geq |S_0| - |S_1|$ edges cut in E_x.

Total edges cut:
$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$
$|V_0| = |V|/2 \geq |S|$.
Also, case 3 where $|S_1| \geq |V|/2$ is symmetric. □

Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0,1\}^n$.

Central area of study in computer science!

Yes/No Computer Programs $\equiv$ Boolean function on $\{0,1\}^n$

Central object of study.