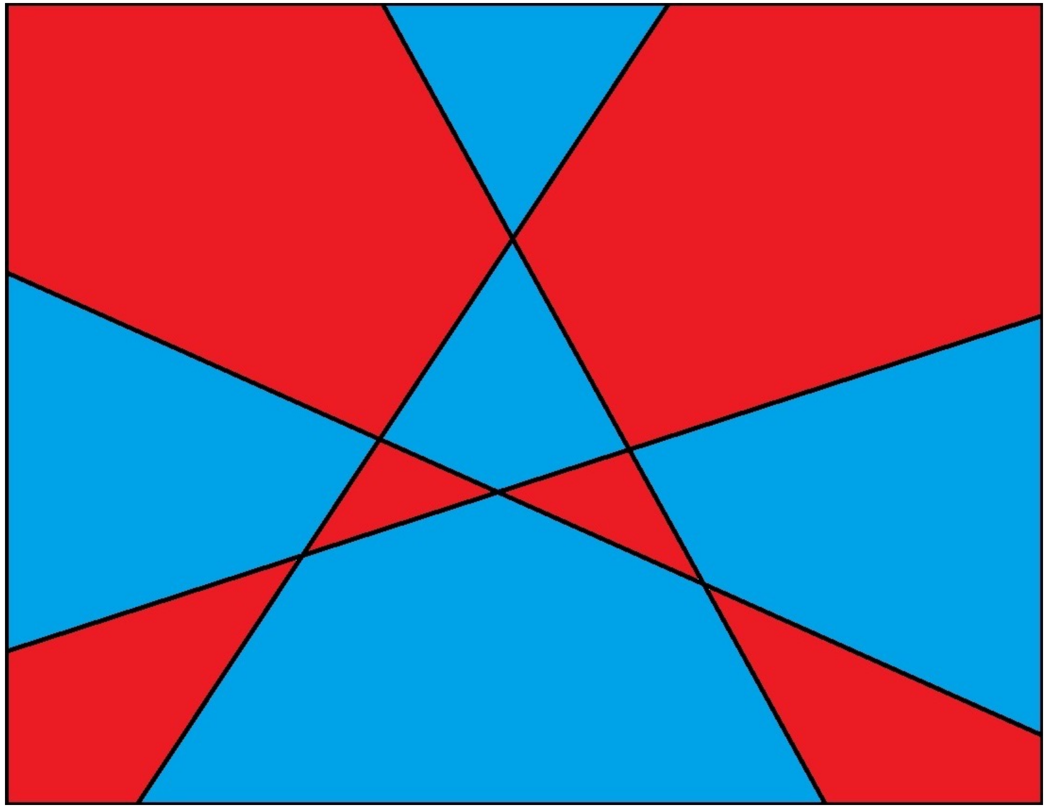
Lecture 3: Proof by Induction



Recap of Lecture 2

Structure of a mathematical proof

A mathematical proof is many lines of propositions

proposition-1
proposition-2
proposition-3
.
.
.
proposition-n (the conclusion we want to prove)

Each line is either known to be correct / derived from previous lines.

Recap of Lecture 2

What make the proof valid.

First, lines that are known to be correct are correct. ✅

Second, lines that derived from known-to-be correct lines are correct. ✅

Third, lines that derived from known-to-be correct lines and lines that became correct in the second step are correct. ✅

.

.

.

.

At last, the conclusion becomes correct. ✅

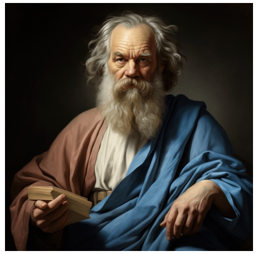
Recap of Lecture 2

- The art of writing mathematical proofs.
  - Direct Proof.
  - Proof by contraposition.
  - Proof by contradiction.
  - Proof by cases.

Today's Plan

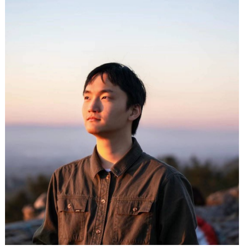
- Proof by induction.
  - The true definition of natural numbers.
  - What is induction?
- Examples.
  - Gauss summation
  - Two coloring theorem
  - Perfect square (strengthening hypothesis)

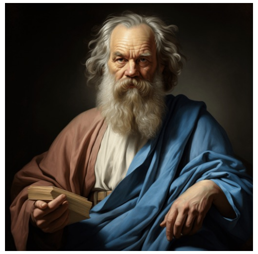
The true definition of natural numbers



What are natural numbers?

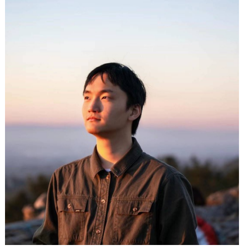
0,1,2,3, …





Meaningless! What is 0?
What is 1? What is 2?
Why is it not 0, 2, 3, 1, ..?

Umm.....



The true definition of natural numbers

Definition.

First, we define that 0 is a natural number. (Fiat Lux! – This is how you create a world.)

Then, we define a successor of 0 to be different from 0, and call it 1.

Then, we define a successor of 1 to be different from 0 or 1, and call it 2.

……………

Structure of natural numbers.

$$0 \xrightarrow{+1} 1 \xrightarrow{+1} 2 \xrightarrow{+1} \cdots\cdots\cdots\ n \xrightarrow{+1} n+1 \longrightarrow \cdots\cdots\cdots$$

Every natural number has to be reachable from 0 in finite steps!
But the total number of natural numbers is infinite.

For example, the number n is reachable in n steps.

What is induction?

Example

Show that $n \leq 2^n$ for all natural number n.

Direct Proof

Define P(n) = the proposition $n \leq 2^n$.

P(0) is true because $0 \leq 1 = 2^0$.

P(0) $\Longrightarrow$ P(1)

P(1) $\Longrightarrow$ P(2)

………….

P(n) $\Longrightarrow$ P(n+1)

………….

This will be an infinitely long proof.

What is induction?

Example

Show that $n \leq 2^n$ for all natural number n.

Induction Proof

Define P(n) = the proposition $n \leq 2^n$.

P(0) is true because $0 \leq 1 = 2^0$.

P(0) $\Longrightarrow$ P(1)

P(1) $\Longrightarrow$ P(2)

....

P(n) $\Longrightarrow$ P(n+1)

....

Let's prove all these in one shot.

For any integer n, suppose P(n) is true ($n \leq 2^n$).

Then $2^{n+1} = 2^n + 2^n \geq n + n \geq n + 1$.

Thus P(n + 1) is true.

~~This will be an infinitely long proof.~~

What is induction?

Example

Show that $n \leq 2^n$ for all natural number n.

Induction Proof

Define P(n) = the proposition $n \leq 2^n$.

P(0) is true because $0 \leq 1 = 2^0$.

Base Case

Induction Hypothesis

For any integer n, suppose P(n) is true ($n \leq 2^n$).

Then $2^{n+1} = 2^n + 2^n \geq n + n \geq n + 1$.

Thus P(n + 1) is true.

Inductive Step

Thus, P(n) is true for all natural number.

What is induction?

Structure of an Induction Proof.

Objective: Prove P(n) holds for all natural number n.
Approach:

First, Prove P(0) is true. — Base Case

Then, assume P(n) is true. — Induction Hypothesis
Prove that P(n + 1) is true. — Inductive Step

Thus, P(n) is true for all natural number.

What makes an induction proof valid?

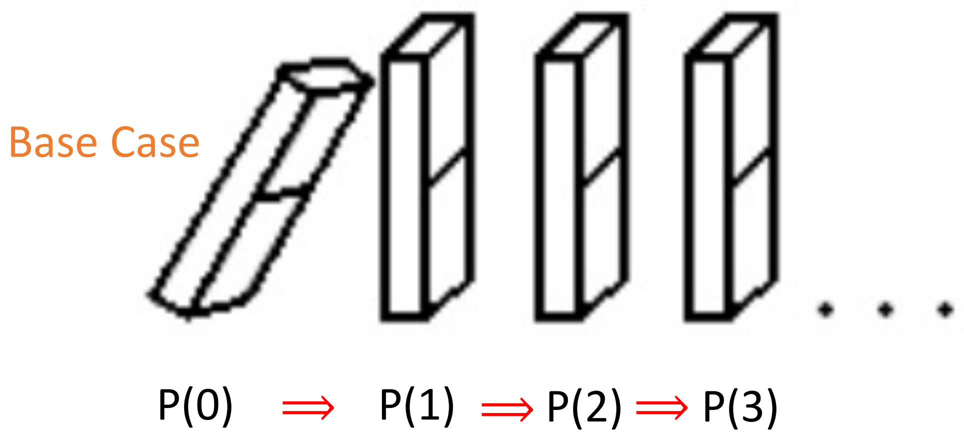
Structure of natural numbers.

0 —(+1)→ 1 —(+1)→ 2 —(+1)→ …………… n —(+1)→ n+1 → ……………

Verify an induction proof.

✅ P(0) —(Inductive step)→ ✅ P(1) —(Inductive step)→ ✅ P(2) —(Inductive step)→ …………… ✅ P(n) —(Inductive step)→ ✅ P(n+1) → ……………

Visual Illustration



Analogy with programing

Direct Proof

Define P(n) = the proposition $n \leq 2^n$.

P(0) is true because $0 \leq 1 = 2^0$.

P(0) $\Longrightarrow$ P(1)

P(1) $\Longrightarrow$ P(2)

…………

P(n) $\Longrightarrow$ P(n+1)

…………

This will be an infinitely long proof.

Induction Proof

First, Prove P(0) is true.

Then, assume P(n) is true.

Prove that P(n + 1) is true.

Thus, P(n) is true for all natural number.

Direct Program printing 0,1,2,….,n

```
print(0)
print(1)
print(2)
………
print(n)
```

This will be a very long program.

For loop Program

```
For i = 0 …. n
    print(i)
```

Take out the heart of the program.

Make it universal for all i.

Today's Plan

- Proof by induction.
  - The true definition of natural numbers.
  - What is induction?

- Examples.
  - Gauss summation
  - Two coloring theorem
  - Perfect square (strengthening hypothesis)

Gauss Summation

This guy

- When Gauss was 7 years old…..
  - Teacher : Hello class
  - Teacher: Please add the numbers from 1 to 100.

  - Gauss: It's 5050! (= 100 * 101 / 2)
  - Teacher: ?????

- Gauss Summation
  - For any natural number n, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Gauss Summation

This guy

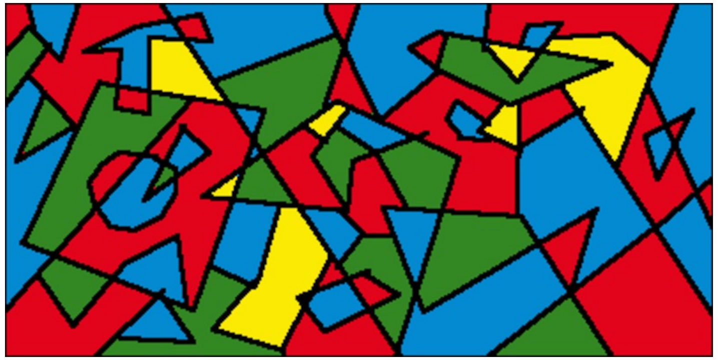
- Proof by induction.
  - Let P(n) be $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
  - Base case: P(0) is true because 0 = 0.
  - Induction Hypothesis: Suppose P(n) is true.
  - Inductive Step:

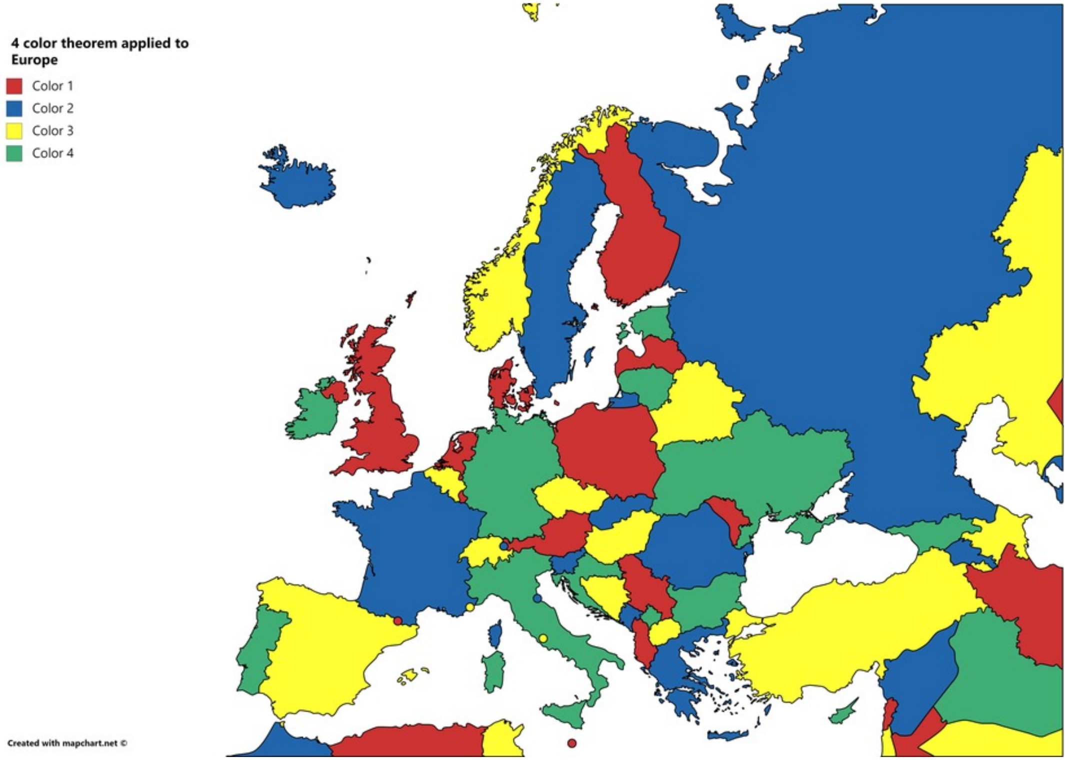
    $1 + 2 + 3 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$.

    Thus P(n + 1).

Four coloring theorem

Theorem.
The regions on any map can be colored using four colors such that no adjacent regions have the same color.





Four coloring theorem

Theorem.
The regions on any map can be colored using four colors such that no adjacent regions have the same color.

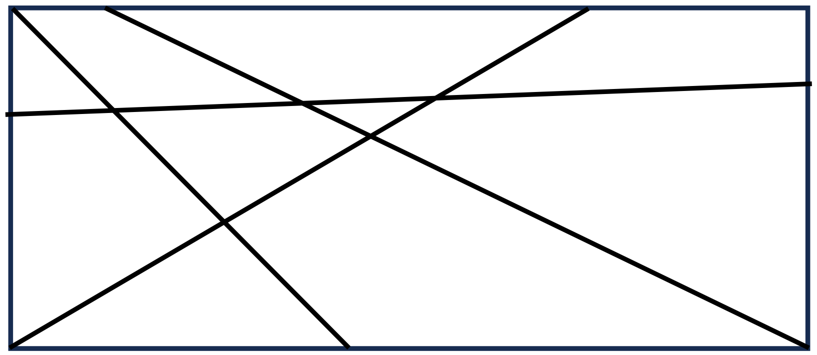
Notoriously hard problem in math!

Took ~100yrs to have a computer-assisted proof (Appel & Haken, 1976).

I bet it has an elegant proof. But maybe need another 100yrs to be discovered.

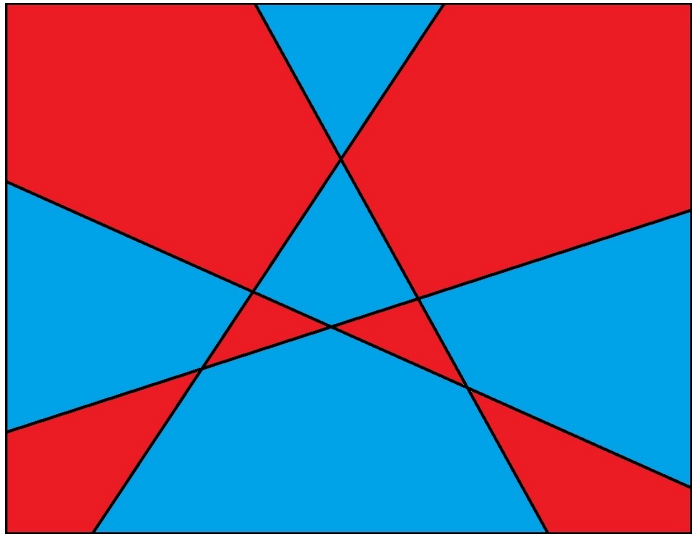
Two coloring theorem

Theorem.
When our map is generated by drawing straight lines cutting a rectangle.
The regions on our map can be colored using two colors such that no adjacent regions have the same color.

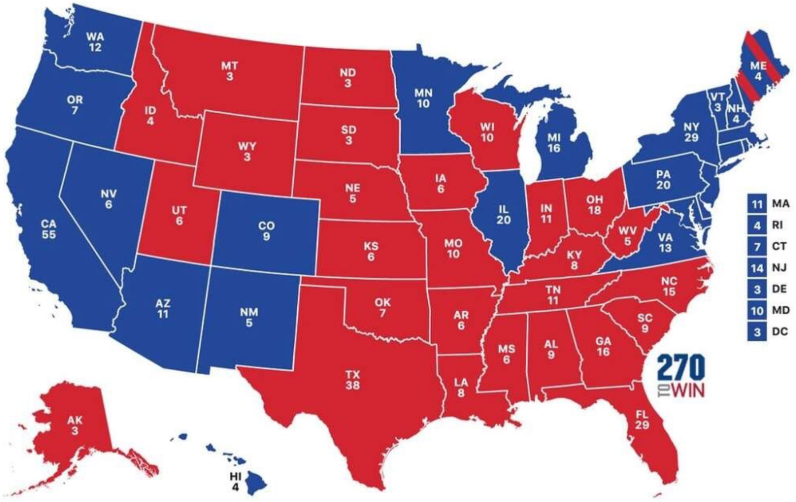


Our map

Two coloring theorem



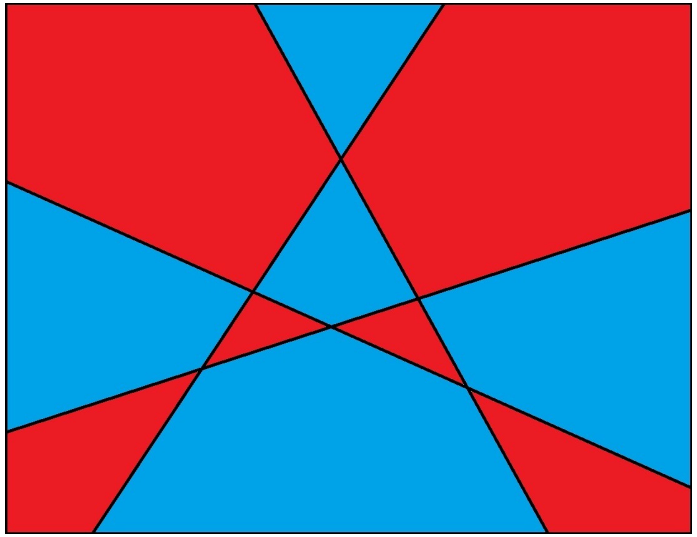
What is a valid two coloring



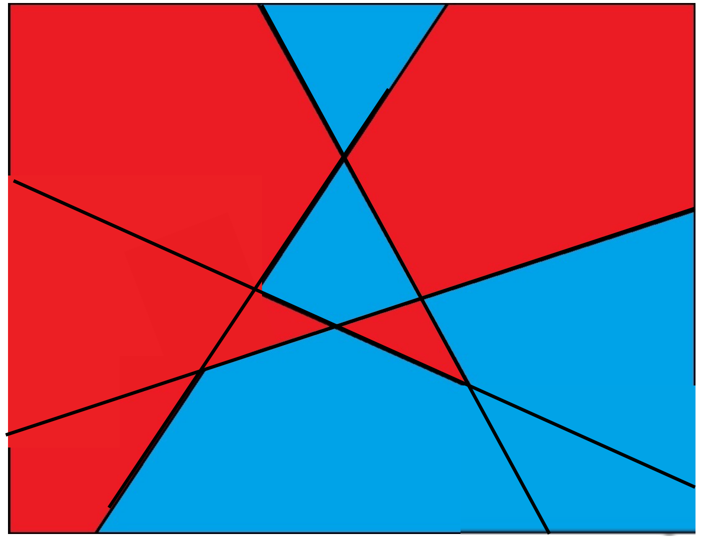
What is NOT a valid two coloring

Instead, it is the result of the 2020 election…

Two coloring theorem



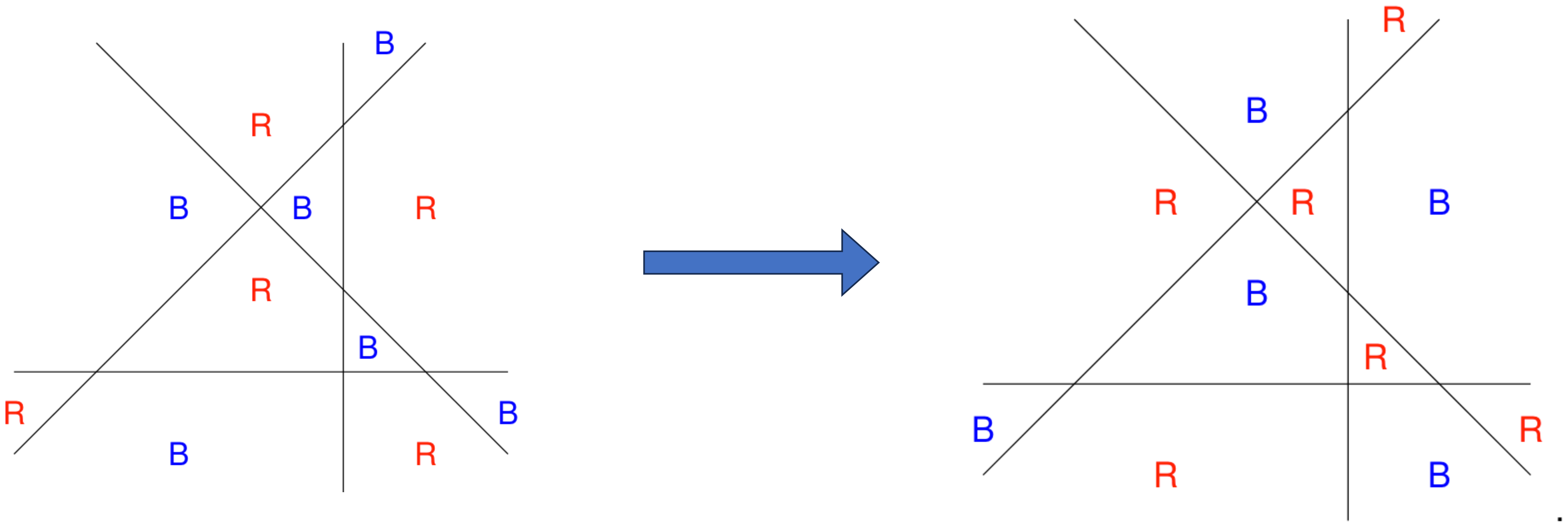
What is a valid two coloring



What is NOT a valid two coloring

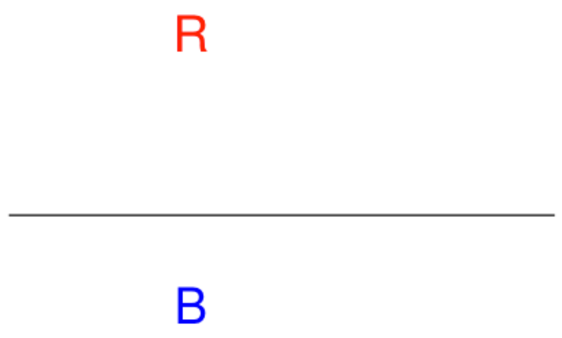
Two coloring theorem

- Fact we will need
  - Swapping blue and red gives another valid coloring.



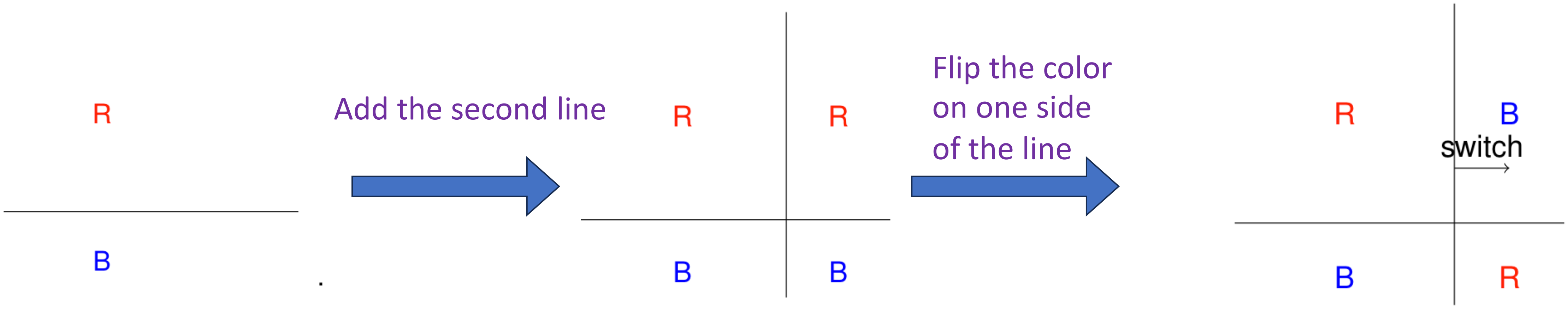
Two coloring theorem

- Proof by induction.
  - Let P(n) be any map formed by n straight lines can be two colored .
  - Base case: P(1) is true because we can color it with two colors.



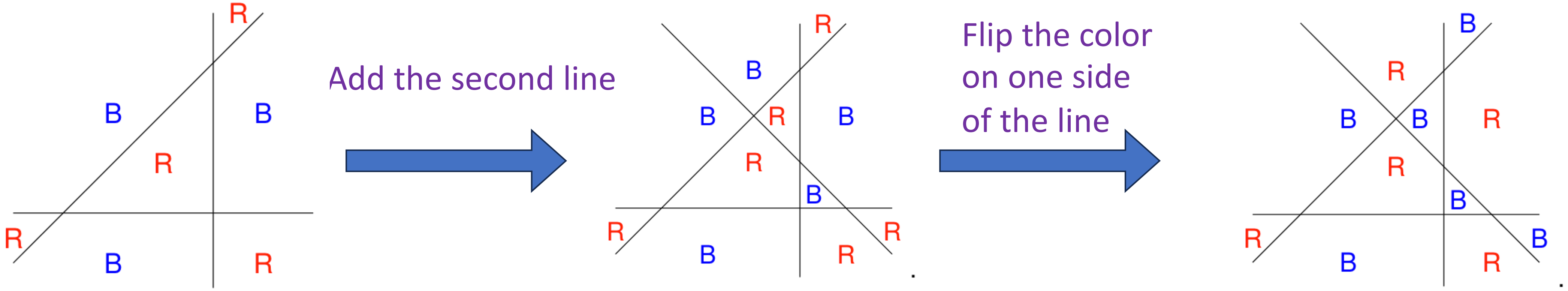
Two coloring theorem

- Proof by induction.
  - Let P(n) be any map formed by n straight lines can be two colored .
  - Base case: P(1) is true because we can color it with two colors.
  - Induction hypothesis: Suppose P(n) is true.
  - Inductive step:
    - Let's first see how to prove P(1) => P(2).



Two coloring theorem

- Proof by induction.
  - Let P(n) be any map formed by n straight lines can be two colored .
  - Base case: P(1) is true because we can color it with two colors.
  - Induction hypothesis: Suppose P(n) is true.
  - Inductive step:
    - Let's first see how to prove P(n+1).



Perfect Square (strengthening hypothesis)

Theorem.
Sum of the first n odd number is a perfect square.

- Proof by induction.
  - Let P(n) be sum of the first n odd number is n^2.
  - Base case: P(1) is true because $1 = 1^2$.
  - Induction Hypothesis: Suppose P(n) is true.
  - Inductive Step:
    $1 + 2 + \ldots. + (2n - 1) + (2(n+1) - 1) = n^2 + 2(n+1) - 1 = (n+1)^2$
    Thus P(n+1) is true.

- Moral of the story:
  - In a direct proof, proving a stronger statement P(n) is harder.
  - In an induction proof, we need to prove P(n) => P(n + 1).
    - P(n+1) is stronger, which makes the proof harder.
    - But P(n) is also stronger, which makes the proof easier.
    - Overall, proving P(n) => P(n+1) for a stronger P(n) may be easier!