Counting – Part 1

CS70: Discrete Mathematics and Probability Theory

UC Berkeley – Summer 2025

Lecture 12

Ref: Note 10

Probability

Wait! Probably doesn't come until later! A little preview:

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide.

Today: Counting!

Later: Probability.

A Look Into The Future!

What's to come? Probability.

A bag contains:

What is the chance that a ball taken from the bag is blue?

Count blue. Count total. Divide. **Chances?**

(A) Red probability is $3/8$? Yes!
(B) Blue probability is $3/9$? No...
(C) Blue probability is $3/8$? Yes!
(D) Yellow probability is $2/8$? Yes!

You can all count!

Combinatorics is fancy counting.

Outline: Counting Basics

1. Counting
2. Tree of Choices
3. Rules of Counting
4. Sample with/without replacement where order does/doesn't matter

What Might You Count?

How many outcomes possible for k coin tosses?

How many poker hands?

How many handshakes for n people?

How many diagonals in a n sided convex polygon?

How many 10 digit numbers?

How many 10 digit numbers without repetition?

How many ways can I divide up 5 dollars among 3 people?

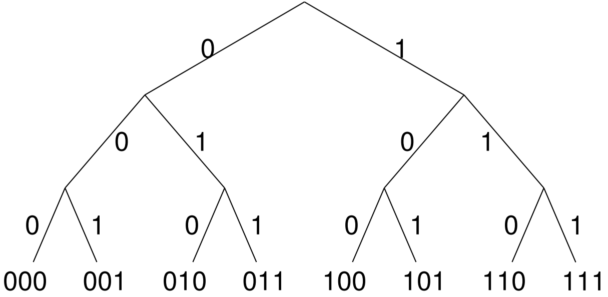
Using a Tree

How many 3-bit binary strings?

How many different sequences of three bits from $\{0, 1\}$?

How would you make one sequence?

How many different ways to do that making?

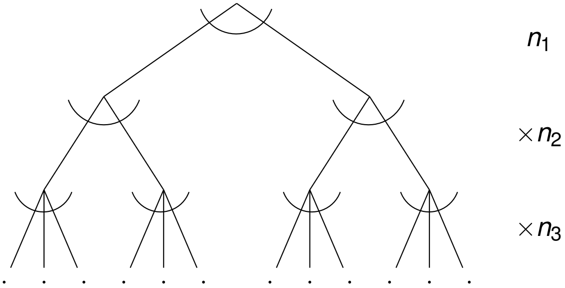


8 leaves which is $2 \times 2 \times 2$. One leaf for each string.

$\Rightarrow$ 8 3-bit strings!

First Rule of Counting: Product Rule

Objects made by choosing from n_1, then from n_2, ..., then from n_k
... the number of objects is $n_1 \times n_2 \cdots \times n_k$.



In picture, $2 \times 2 \times 3 = 12$

Using The First Rule

How many outcomes possible for k coin tosses?

2 ways for first choice, 2 ways for second choice, ...

$2 \times 2 \times \cdots \times 2 = 2^k$

How many 10 digit numbers?

10 ways for first choice, 10 ways for second choice, ...

$10 \times 10 \times \cdots \times 10 = 10^{10}$

How many n digit base m numbers?

m ways for first, m ways for second, ...

$m \times m \times \cdots \times m = m^n$

But: Is 09, a two digit number?

If not, then $(m-1)m^{n-1}$

Counting Functions and Polynomials

How many functions f mapping S to T? (i.e., $f : S \to T$)

$|T|$ ways to choose for $f(s_1)$, $|T|$ ways to choose for $f(s_2)$, ...

... $|T|^{|S|}$

How many polynomials of degree d modulo p?

p ways to choose for first coefficient, p ways for second, ...

... p^{d+1}

Counting by points at $x = 0, 1, 2, ..., d$ rather than coefficients ($d + 1$ points!):

p values for first y value, p values for second, ...

... p^{d+1}

Questions?

Permutations

How many 10 digit numbers **without repeating a digit**?

10 ways for first, 9 ways for second, 8 ways for third, ...

... $10 \times 9 \times 8 \times \cdots \times 1 = 10!$

Sample of size k from n numbers **without replacement**?

n ways for first choice, $n-1$ ways for second, $n-2$ choices for third, ...

... $n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$

How many orderings of n objects? (**Permutations of n objects**)

n ways for first, $n-1$ ways for second, $n-2$ ways for third, ...

... $n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$

One-to-One Functions

How many one-to-one functions from S to S (bijections on S)?

$|S|$ choices for $f(s_1)$, $|S|-1$ choices for $f(s_2)$, ...

... $|S| \times (|S|-1) \times \cdots 1 = |S|!$

A one-to-one function with the same domain and range is a permutation!

Bijections are permutations – recall:

$f(x) = 2x \pmod 5$ $\quad [0,1,2,3,4] \to [0,2,4,1,3]$

Number of bijections from $\{0,1,2,3,4\}$ to $\{0,1,2,3,4\}$? $5! = 120$

Number of bijections like $f(x) = ax \pmod 5$? 4 (all the non-zero vals for a)

Linear bijections: $f(x) = ax + b \pmod 5$?
4 choices for a (non-zero), 5 choices for b: $4 \times 5 = 20$

$\Longrightarrow$ linear functions (mod p) are a tiny fraction of bijections on $\{0,\ldots,p-1\}$

Counting Sets – When Order Doesn't Matter

How many poker hands? (*52 cards in a deck, all different, 5 in a poker hand*)

$52 \times 51 \times 50 \times 49 \times 48 = \frac{52!}{47!}$???

Are $A, K, Q, 10, J$ of spades
and $10, J, Q, K, A$ of spades the same?

Second Rule of Counting: If order doesn't matter, count ordered objects and then divide by the number of equivalent orderings of an ordered object.

"the" number of equivalent... assumes same for all ordered objects!

Number of orderings for a poker hand: 5!

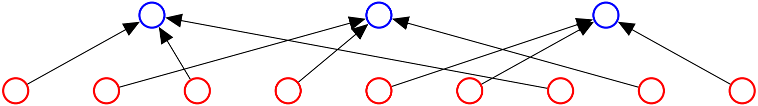
By second rule of counting, number of poker hands is

$$\frac{52!}{47!} \cdot \frac{1}{5!} = \frac{52!}{47! \cdot 5!}$$

Generic: ways to choose 5 out of 52 possibilities.

Ordered to Unordered

Second Rule of Counting: If order doesn't matter count ordered objects and then divide by the number of equivalent orderings.



How many red nodes (ordered objects)? 9

How many red nodes mapped to one blue node? 3

How many blue nodes (unordered objects)? $\frac{9}{3} = 3$

How many poker deals? $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$

How many poker deals per hand?
Map each deal to ordered deal: 5!

How many poker hands? $\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!}$

Questions?

When Order Doesn't Matter

Choose 2 out of n?

$$\frac{n \times (n-1)}{2} = \frac{n!}{(n-2)! \times 2}$$

Choose 3 out of n?

$$\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{(n-3)! \times 3!}$$

Choose k **out of** n?

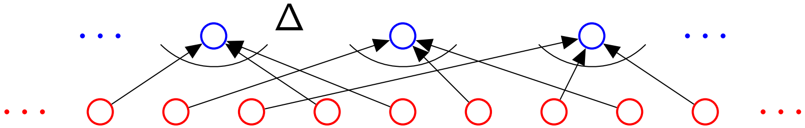
$$\frac{n!}{(n-k)! \times k!}$$

Notation: $\binom{n}{k}$ **and pronounced "n choose k"**

Familiar? Questions?

Example: Visualize the Proof

k choices. *First Rule:* $n_1 \times \cdots \times n_k$
Second Rule: when order doesn't matter, divide...



3 card poker deals: $52 \times 51 \times 50 \;\; (= \frac{52!}{49!})$. *First Rule*

Poker hands: Δ?

Hand: Q, K, A.

Deals: $Q,K,A : Q,A,K : K,A,Q : K,A,Q : A,K,Q : A,Q,K$

$\Delta = 3 \times 2 \times 1$ *First Rule again*

Total: $\frac{52!}{49!3!}$ *Second Rule!*

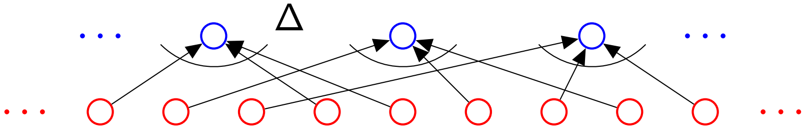
Choose k out of n:

Ordered set: $\frac{n!}{(n-k)!}$ (*First Rule*) Orderings of one hand? $k!$ (*First Rule*)

$\Longrightarrow$ Total: $\frac{n!}{(n-k)!k!}$ (*Second rule*)

Example: Anagrams

k choices. *First Rule:* $n_1 \times \cdots \times n_k$
Second Rule: when order doesn't matter, divide...



Orderings of ANAGRAM?

Ordered Set: 7! (*First Rule*)

A's are the same! What is Δ?

ANAGRAM

$A_1NA_2GRA_3M$, $A_2NA_1GRA_3M$, ...

$\Delta = 3 \times 2 \times 1 = 3!$ (*First Rule*)

$\Longrightarrow$ Total $\frac{7!}{3!}$ (*Second Rule*)

Concept Check

For each of these: correct or not?

(A) Number of poker hands $= \binom{52}{5}$ Yes!

(B) Orderings of "CAT" $= 3!$ Yes!

(C) Orderings of ANAGRAM $= \frac{7!}{3!}$ Yes!

(D) Orderings of ANAGRAM $= \frac{7!}{4!}$ No!

(E) Orders of MISSISSIPPI $= \frac{11!}{4!4!2!}$ Yes!

(F) Orders of MISSISSIPPI $= \frac{11!}{10!}$ No!

Explanations

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways for second, 1 for last.

$\implies 3 \times 2 \times 1 = 3!$ orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A

Total orderings of 7 letters $= 7!$
Total equivalent orderings of three A's $= 3!$

Total orderings? $\frac{7!}{3!}$

How many orderings of MISSISSIPPI?

11 letters total: 4 S's, 4 I's, 2 P's

$11!$ ordered objects.
$4! \times 4! \times 2!$ ordered objects per "unordered object"

Total orderings? $\frac{11!}{4!4!2!}$

Important: Not just 10 letters rearranged – 3 groups of letters

Choose order of S's (4!), choose order of I's (4!), choose order of P's (2!). Multiply (*First Rule*)

How To Sample k Items Out Of n

Without replacement:

Order matters: $n \times (n-1) \times (n-2) \ldots \times (n-k+1) = \frac{n!}{(n-k)!}$

Order does not matter:

Second Rule: divide by number of orders – "$k!$"

$\Longrightarrow \frac{n!}{(n-k)!k!}$ or "n choose k" or $\binom{n}{k}$

With Replacement:

Order matters: $n \times n \times \ldots n = n^k$

Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!

Different number of ordered items map to each unordered item

Unordered item: $1,2,3$ $3!$ ordered items map to it

Unordered item: $1,2,2$ $\frac{3!}{2!}$ ordered items map to it

How do we deal with this mess??

Splitting Up Money: The Problem

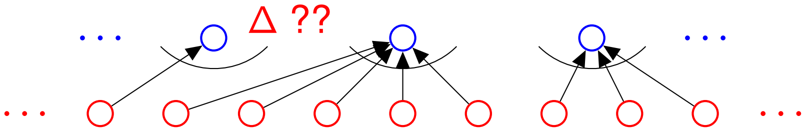
Problem: How many ways can Bob and Alice split 5 dollars?

For each of 5 dollars pick Bob or Alice (2^5), divide out order???

5 dollars for Bob and 0 for Alice:
  one ordered set: (B, B, B, B, B)

4 for Bob and 1 for Alice:
  5 ordered sets: (A, B, B, B, B) ; (B, A, B, B, B); ...

"Sorted" way to specify, first Alice's dollars, then Bob's.
  (B, B, B, B, B): 1: (B,B,B,B,B)
  (A, B, B, B, B): 5: (A,B,B,B,B), (B,A,B,B,B), (B,B,A,B,B),...
  (A, A, B, B, B): $\binom{5}{2}$: (A,A,B,B,B), (A,B,A,B,B), (A,B,B,A,B), ...
  and so on.



Second rule of counting is no good here!

Splitting Up Money: Stars and Bars

Problem: How many ways can Alice, Bob, and Eve split 5 dollars?

One way: Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E)

Idea: Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: $\star\star\star\star\star$
Separate with bars: Two "middle separators" (bars) for three sets.
$(A, A, A, B, E) \longrightarrow$ Stars and Bars: $\star\star\star|\star|\star$.

Alice: 2, Bob: 1, Eve: 2.
$\longrightarrow$ Stars and Bars: $\star\star|\star|\star\star$.

Alice: 0, Bob: 1, Eve: 4.
$\longrightarrow$ Stars and Bars: $|\star|\star\star\star\star$.

Each split "is" a sequence of stars and bars
... and each sequence of stars and bars "is" a split
... a bijection!

Counting Rule: If there is a bijection between two sets they have the same size!

Counting Stars and Bars

Question: How many different 5 star and 2 bar diagrams?

Regardless of how split: 7 symbols – 5 are stars, 2 are bars
To help visualize: Number the symbol positions 1 to 7

Alice: 0; Bob 1; Eve: 4

| $\star$ | $\star$ $\star$ $\star$ $\star$ Bars in positions 1 and 3

1 2 3 4 5 6 7

Alice: 1; Bob 4; Eve: 0

$\star$ | $\star$ $\star$ $\star$ $\star$ | Bars in positions 2 and 7

1 2 3 4 5 6 7

Picking 2 positions for bars ... out of 7 positions ...

$\binom{7}{2}$ ways to pick $\longrightarrow$ $\binom{7}{2}$ ways to split 5 dollars among 3 people.

Stars and Bars

Back to general problem:

Sample k from n with replacement, where order doesn't matter
$\Rightarrow$ Only the count of each of the n items matters

k selections (stars) ; separated into n groups with $n-1$ bars

$$\star\star|\star|\cdots|\star\star$$

k stars, $n-1$ bars $\longrightarrow$ $n+k-1$ length string

Select $n-1$ of the $n+k-1$ positions for the bars

Ways to do so:

$$\binom{n+k-1}{n-1}$$

Could also select positions for the k stars – gives the same count:

$$\binom{n+k-1}{k}$$

Concept Check

For each of these: correct or not?

(A) Ways to split k dollars among n: $\binom{k+n-1}{n-1}$ Yes!

(B) Ways to split n dollars among k: $\binom{n+k-1}{k-1}$ Yes!

(C) Ways to split 5 dollars among 3: $\binom{5+3-1}{3-1}$ Yes!

(D) Ways to split 5 dollars among 3: $\binom{7}{5}$ Yes!

All correct! ... Would I tell you something untrue?
On a true/false question? Yes, yes I would.

Summary

First rule: $n_1 \times n_2 \times \cdots \times n_k$

k Samples with replacement from n items: n^k.

Sample without replacement: $\frac{n!}{(n-k)!}$

Second rule: when order doesn't matter divide .. when possible

Sample without replacement, no order: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$. "$n$ choose k"

One-to-one rule: equal in number if one-to-one correspondence

Sample with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$

Distribute k samples (stars) over n poss. ($n-1$ bars group poss..)

Distribute k dollars to n people.