

Countability

CS70: Discrete Mathematics and Probability Theory

UC Berkeley – Summer 2025

Lecture 14

Ref: Note 11

Today

Sizes of sets

- Comparing sizes of two sets – bijections

- Infinite sets too!

How large are the common sets?

- Natural Numbers?

- Integers?

- Rationals?

- Reals?

Other infinite sets?

- Set of finite-length binary strings?

- Set of subsets of natural numbers?

Along the way:

- A new proof technique: diagonalization

Initial Poll: Sizes of Infinite Sets

Question: Which of the following are true?

- (A) There are more real numbers than natural numbers.
- (B) There are more integers than natural numbers.
- (C) There are more rational numbers than natural numbers.
- (D) There are more pairs of natural numbers than natural numbers.

Why? ... and how do these even make sense?

Comparing Apples to Oranges

Back to Kindergarten – counting sets of fruit!

Apples = { Braeburn, McIntosh, Gala, Fuji }



Oranges = { Valencia, Mandarin, Blood, Navel }

Can count each set: 4 elements each
Same size!

From Lecture 12:

Counting Rule: If there is a bijection between two sets they have the same size!

... so:

Bijection between sets, so the same size

(*maybe you didn't learn bijections in kindergarten...*)

Can we do this with *infinite* sets?

Counting

How to count?

Counting: 0, 1, 2, 3, ...

We're counting stuff... what if we don't have any stuff?

So we need zero too

The natural numbers: $\mathbb{N}$

Definition: A set S is **countable** if there is a bijection between S and some subset of $\mathbb{N}$.

If the subset of $\mathbb{N}$ is finite, S has finite **cardinality**.

If the subset of $\mathbb{N}$ is infinite, S is **countably infinite**.

Infinity Plus One

Back to grade school....

Bart: I've got 100 cookies

Lisa: Oh yeah, I've got 200 cookies!

Bart: I've got infinity cookies!!

Lisa: Oh yeah?!? I've got infinity plus one!!!! Game over!

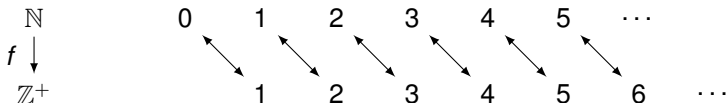
A little more math-y:

$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ Size? Infinity?

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ Everything in $\mathbb{Z}^+$ – and zero. Infinity plus one?

Bijection $f : \mathbb{N} \rightarrow \mathbb{Z}^+$

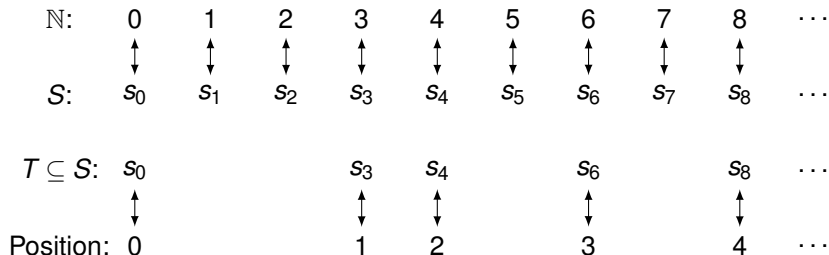
$$f(x) = x + 1$$



Same size! (?!?!)

Subsets of Countable Sets

What about subsets of a countable set?



Theorem: If S and T are infinite, with $T \subseteq S$ and S is countably infinite, then T is countably infinite.

Note:

We don't need to know the “position” of each element in T

The position doesn't need to be *computable*

It's enough that it *exists*

Countability of $\mathbb{Z}^+$: We have $\mathbb{Z}^+ \subseteq \mathbb{N}$, so $\mathbb{Z}^+$ is countable.

Integers

$\mathbb{Z}$ has both positive and negative values – must be bigger than $\mathbb{N}$, right?

No! Interleaving bijection:

$\mathbb{N}$:	0	1	2	3	4	5	6	7	8	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Z}$:	0	-1	1	-2	2	-3	3	-4	4	...

Theorem: $\mathbb{Z}$ is countable.

Enumeration Example: Finite Length Binary Strings

Notation:

$B = \{0, 1\}^*$ is the set of all (finite length) binary strings

ε is the empty string

Enumeration: All length 0, then all length 1, then all length 2, ...

0: ε

Are all strings on the list?

1: 0

Yes!

2: 1

String $b \in \{0, 1\}^*$

3: 00

Finitely many *shorter* strings

4: 01

Finitely many *of the same length*

5: 10

$\Rightarrow$ Finite position on list

6: 11

7: 000

Can even calculate position:

8: 001

Take strings b , prepend a 1

9: 010

Treat as a binary number and subtract 1

...

Try some!

Theorem: The set of all finite length binary strings is countable.

Pairs of Numbers

Consider $S = \{1, 2, 3\}$

$S \times S$ is the set of *pairs* from S

$S \times S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

Called the “Cartesian Product” (of S and S)

Size? $|S \times S| = |S|^2$

What if S is infinite?

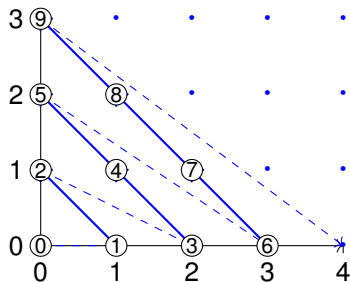
What about $\mathbb{N} \times \mathbb{N}$?

Pairs of natural numbers

$(1, 1), (4, 2), (4, 8), (16326324, 62346124), \dots$

Size? Infinity squared?

Pairs of Natural Numbers



Enumerate list:

$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \dots$

“Sweep d ” ($d = 0, 1, \dots$) hits all (x, y) with $x + y = d$... $(d + 1)$ pairs

Before diagonal d :

Check our work?

$$\sum_{i=0}^{d-1} (i+1) = \frac{d(d+1)}{2}$$

$$f((0,0)) = \frac{0 \cdot 1}{2} + 0 = 0 \checkmark$$

$$f((2,0)) = \frac{2 \cdot 3}{2} + 0 = 3 \checkmark$$

$$f((1,2)) = \frac{3 \cdot 4}{2} + 2 = 8 \checkmark$$

$$f((x,y)) = \frac{(x+y)(x+y+1)}{2} + y$$

Bijection! $\implies$ The set of pairs of natural numbers is countable

Concept Check: Bijections on Sets of Numbers

Question: Which bijection ideas work?

- (A) Integers: First all negatives, then positives
- (B) Integers: By absolute value, break ties however
- (C) Pairs of naturals: by sum of values, break ties however
- (D) Pairs of naturals: by value of first element
- (E) Pairs of integers: by sum of values, break ties
- (F) Pairs of integers: by sum of absolute values, break ties

The Rational Numbers

Consider $S = \{(x, y) \mid x, y \in \mathbb{N} \text{ and } \gcd(x, y) = 1\}$

Each $(x, y) \in S$ corresponds to a fraction $\frac{x}{y}$ in lowest terms.

Each non-negative rational written in lowest terms as $\frac{x}{y}$ and so $(x, y) \in S$.

$\implies$ So bijection between S and $\mathbb{Q}^+$.

We also know that $S \subseteq \mathbb{N} \times \mathbb{N}$

Subset Theorem! We just proved $\mathbb{N} \times \mathbb{N}$ is countable, so S is countable.

$\implies \mathbb{Q}^+$ is countable

What about *all* rational numbers – not just positive ones?

Let $q_0, q_1, q_2, \dots$ be an enumeration of $\mathbb{Q}^+$

Idea: Interleave like we did mapping $\mathbb{N}$ to $\mathbb{Z}$

$\mathbb{N}$:	0	1	2	3	4	5	6	7	8	$\dots$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{Q}$:	0	$-q_0$	q_0	$-q_1$	q_1	$-q_2$	q_2	$-q_3$	q_3	$\dots$

Theorem: $\mathbb{Q}$ is countable.

The Reals

Is the set of reals $\mathbb{R}$ countable?

Lets consider the reals $[0, 1]$.

Each real has a decimal representation

.500000000... $(1/2)$

.333333333... $(1/3)$... possibly with infinite non-zero digits

.785398162... $\pi/4$... possibly with infinite non-repeating digits

.367879441... $1/e$

.632120558... $1 - 1/e$

.345212312... Some number ... possibly no pattern or meaning at all

Countable?

Can we make a numbered list of reals in $[0, 1]$?

Diagonalization

Assume for the sake of contradiction there's a mapping $\mathbb{N} \rightarrow \mathbb{R}[0, 1]$:

0: .500000000...
1: .333333333...
2: .785398162...
3: .367879441...
4: .632120558...
5: .345212312...
⋮

Construct “diagonal number” – digits from diagonal, add 2 (mod 10) to each:

.757044 ...

Can the diagonal number be in the list?

Position n ? No! Digit $n+1$ differs

Subtle point: Why add *two*? avoids problems like $0.25 = 0.2499999...$

Diagonal number is a real number

Diagonal number is not in the list

The list is a list of all real numbers

Contradiction! $\implies \mathbb{R}[0, 1]$ is not countable

All Reals

Could there be less?

Recall the Subset Theorem:

Theorem: If S and T are infinite, with $T \subseteq S$ and S is countably infinite, then T is countably infinite.

Contrapositive: If T is not countable, then S is not countable.

So $\mathbb{R}[0, 1]$ is uncountable $\implies \mathbb{R}$ is uncountable.

All Reals

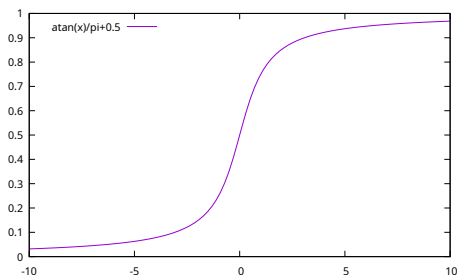
Could there be more?

Showed $\mathbb{R}[0, 1]$ is uncountable – what about all of $\mathbb{R}$?

Is the set of all reals even larger?

No. “Almost” bijection, mapping open $\mathbb{R}$ to open interval $(0, 1)$:

$$f(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}(x)$$



Complete bijection with closed interval $[0, 1]$ is harder...

... but all we care about is cardinality, so 2 points don't change that.

Diagonalization

General outline of a diagonalization proof:

- 1 Assume that a set S can be enumerated
- 2 Consider an arbitrary list of all the elements of S
- 3 Use the diagonal from the list to construct a new element t
- 4 Show that t is different from all elements in the list
 $\implies t$ is not in the list
- 5 Show that t is in S
- 6 Contradiction

Power Sets

$\mathcal{P}(S)$ is the **power set** of S – the set of all subsets of S .

Example:

$$S = \{1, 2, 3\}$$

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Theorem: If S is finite, then $|\mathcal{P}(S)| = 2^{|S|}$.

Proof: Count them!

First element? Choose to include or not. 2 choices.

Second element? Choose to include or not. 2 choices.

...
 $|S|^{\text{th}}$ element? Choose to include or not. 2 choices.

First rule of counting: $2 \times 2 \times \cdots \times 2 = 2^{|S|}$



What if S is infinite? How big is $\mathcal{P}(S)$?

Another Diagonalization Proof

Theorem: $\mathcal{P}(\mathbb{N})$ is not countable.

Proof: Subset representation: Infinite binary strings

Subset $S \subseteq \mathbb{N}$: Bit i is 1 if and only if $(i - 1) \in S$

$\emptyset = 0000000000000000\dots$

$\{1, 4, 6\} = 01001010000000\dots$

Evens = 1010101010101010...

Primes = 00110101000101...

Assume countable, so enumeration:

0: 111110100111...

1: 100000110101...

2: 111011111010...

3: 000111001000...

4: 011100111100...

...

Make “diagonal set” – flip each bit

01001...

In listing?

At position n :

Bit $(n + 1)$ is flipped

... so can't be at position n

Contradiction!

Concept Check: Review

True or false?

- (A) $\mathbb{Z}$ is larger than $\mathbb{N}$ because it has negatives too
- (B) $\mathbb{Z}$ is countable because of interleaving bijection
- (C) $\mathbb{Q}$ is uncountable because infinitely many between 0 and 1
- (D) Reals in list: “diagonal number” not on list – Contradiction!
- (E) Powerset in list: “diagonal set” not in list

Weirdness

Is anything bigger than $\mathcal{P}(\mathbb{N})$?

What about $\mathcal{P}(\mathcal{P}(\mathbb{N}))$?

Talking about “levels of infinity” – use “aleph ($\aleph$) numbers”

Cardinality of $\mathbb{N}$ is $\aleph_0$

Cardinality of *any countable set* is $\aleph_0$

Cardinality of $\mathcal{P}(\mathbb{N})$ is $2^{\aleph_0}$

Cardinality of $\mathbb{R}$ is $2^{\aleph_0}$

$$\aleph_0 \quad \underbrace{\quad \dots \quad}_{\text{Anything Here?}} \quad 2^{\aleph_0}$$

No ... and yes ...

It's complicated (“Continuum Hypothesis”)

Work of logician Kurt Gödel – old-ish book: *Gödel, Escher, Bach*

A Peak Ahead

How many predicates $P : \mathbb{N} \rightarrow \{\text{True}, \text{False}\}$?

Ex: Is x even? Is x a perfect square? Is x prime? Does Collatz hold for x ?

Same as number of subsets of $\mathbb{N}$ (subset elements $\leftrightarrow$ True values)

$\implies$ Set of predicates is uncountable

How many programs?

A program is a finite length binary string

Set of finite length binary strings is countable

$\implies$ Set of programs is countable

So: More (*many more*) predicates than programs

$\implies$ Programs can't compute all predicates

$\implies$ There are uncomputable functions

Lecture 14 Summary

Sizes of sets

- Comparing sizes of two sets – bijections
- Infinite sets too!

Countably infinite sets

- Counting numbers ($\mathbb{N}$) are countable (surprise!)
- Integers are countable
- Rationals are countable
- The set of all finite-length binary strings is countable

Uncountable sets

- The set of reals is uncountable
- Diagonalization as a proof technique
- The set of subsets of $\mathbb{N}$ is uncountable

Bottom line: Infinity is weird. And cool.