CS70 @ UC Berkeley, Spring 2026

Lecture 14
Counting I

March 5, 2026

Introduction

- The second half of the semester will focus on **probability theory**.
- In discrete probability, we will want to count the number of elements of a finite set.
- Counting such objects is important because, when all outcomes are equally likely, probabilities can be defined as

$$\Pr(A) = \frac{|A|}{|\Omega|}$$

where Ω is the set of possible **outcomes** and $A \subseteq \Omega$ is a sebset of outcomes satisfying some property.

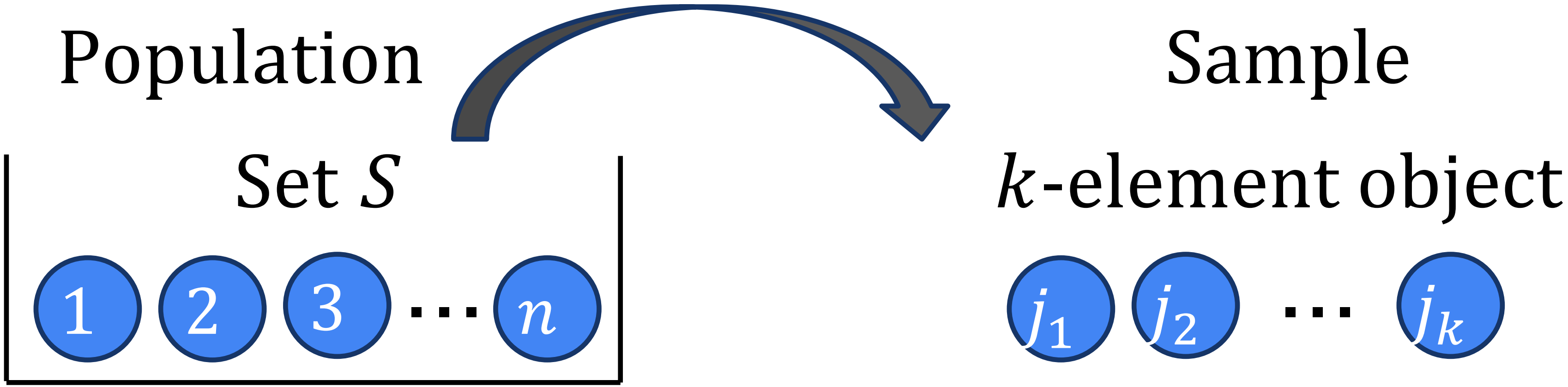
Introduction

- However, counting can quickly become **surprisingly subtle** and **difficult**.
- Many important objects in mathematics are **defined implicitly**.
- Seemingly small changes in constraints can dramatically change the count.
  - E.g., counting unconstrained functions $f: X \to S$ is relatively easy, but counting functions subject to certain restrictions is more involved.
  - Often we must count objects that satisfy several simultaneous constraints.
- **Direct counting is often infeasible**; e.g., some sets are extremely large.

Introduction

- Counting often requires **clever reasoning** rather than brute force.
- **Counting the same set in different ways** can reveal deep mathematical identities.
- **Techniques** include **bijections, inclusion-exclusion,** generating functions, and recursion. We will learn some of these techniques in this course.
- Developing good counting techniques is essential for studying discrete probability and many areas of computer science, statistics, and mathematics.

Sampling



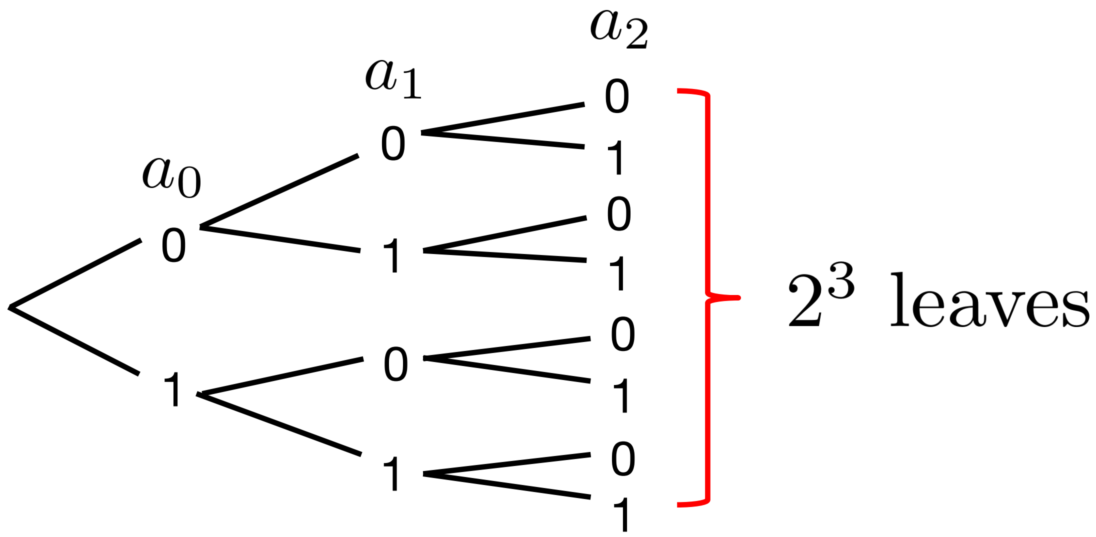
\# **inequivalent** k-element objects depends on whether

- sampled objects are **<u>ordered</u>** or **unordered**
- sampling is done **<u>with</u>** or without **replacement**

Case 1: Ordered objects sampled with replacement

- #distinct polynomials over GF(2) with degree ≤ 2?

$$P(x) = a_2 x^2 + a_1 x + a_0, \text{ where } a_0, a_1, a_2 \in \text{GF}(2)$$



$S = \{0,1\}$

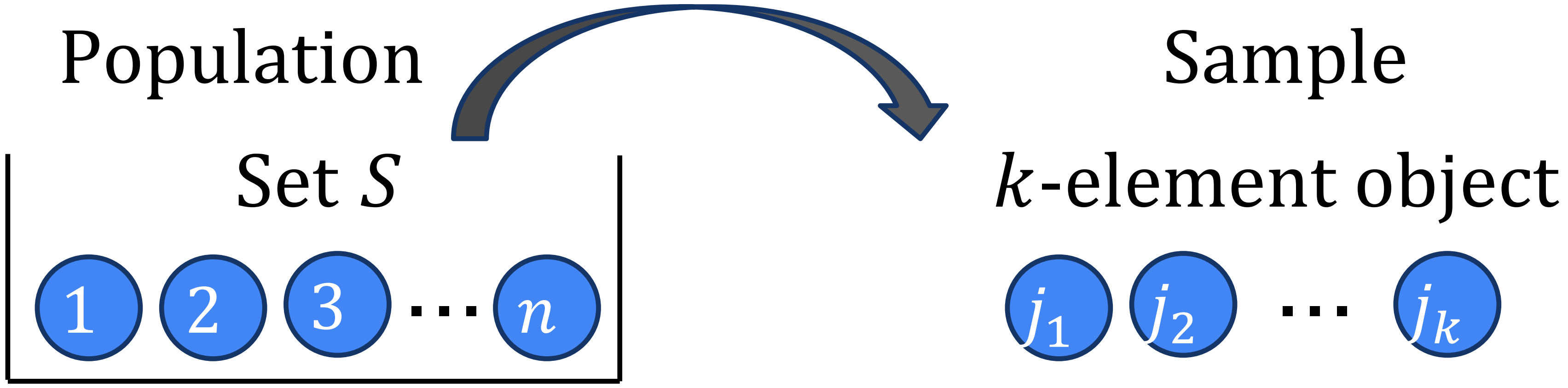
Each path from the root to a leaf defines a unique polynomial over GF(2)

- #distinct polynomials over GF(n) with degree ≤ $k-1$?

$$P(x) = a_{k-1} x^{k-1} + \cdots + a_1 x + a_0, \text{ where } a_0, a_1, \ldots, a_{k-1} \in \text{GF}(n)$$

Answer: n^k $\qquad S = \{0,1,\ldots,n-1\}$

Sampling

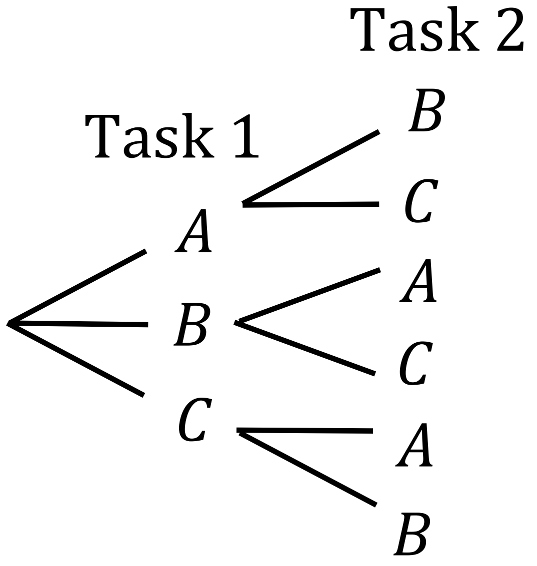


inequivalent k-element objects depends on whether

- sampled objects are **ordered** or **unordered**
- sampling is done with or **without** **replacement**

Case 2: Ordered objects sampled without replacement

ways to distribute 2 tasks to 3 compute nodes $\{A, B, C\}$, with at most 1 task per node. $S = \{A, B, C\}$



$3 \times 2 = 6$ leaves

General case: # ways to distribute k tasks to n compute nodes, with at most 1 task per node.

distinct length-k strings

$$= n(n-1)(n-2)\cdots(n-k+1)$$

$$= \frac{n!}{(n-k)!}$$

"kth falling factorial of n"

If $k = n$, then this expression becomes $n!$, which is the number of **permutations** on n distinct elements.

First Rule of Counting

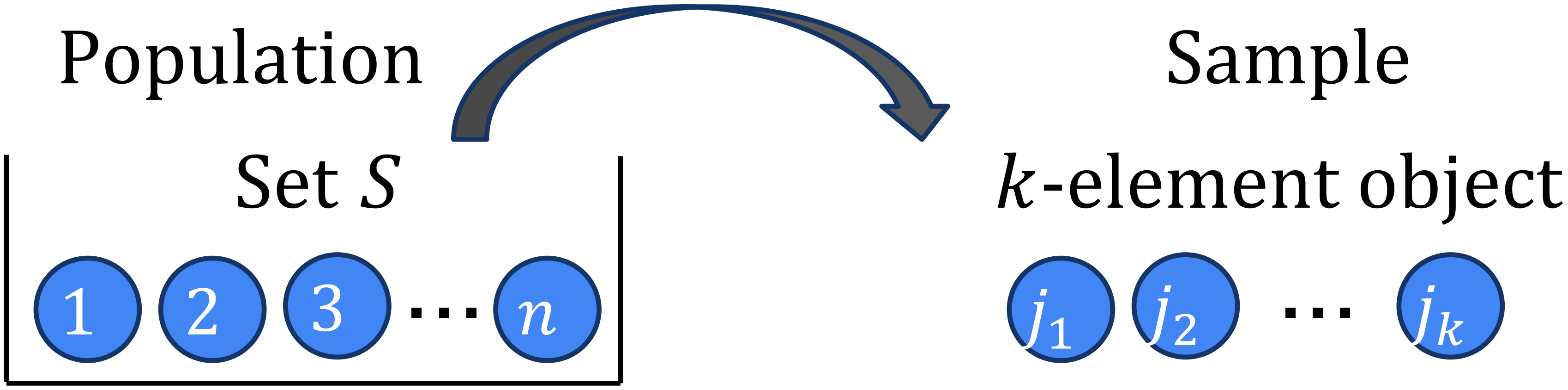
Suppose an ordered object is formed by a succession of k choices where there are

n_1 possibilities for the 1st choice,
n_2 possibilities for the 2nd choice,
$\vdots$
n_k possibilities for the k^{th} choice.

Then, the total number of distinct ordered objects is given by

$$n_1 \times n_2 \times \cdots \times n_k.$$

Sampling



inequivalent k-element objects depends on whether

- sampled objects are **ordered** or <u>**unordered**</u>
- sampling is done with or <u>**without**</u> **replacement**

Counting functions between two finite sets

$$f: X \to S, \text{ where } |X| = k \text{ and } |S| = n$$

The number of inequivalent functions f depends on:

Case 1: f arbitrary; both X and S contain distinguishable elements

Case 2: f injective; both X and S contain distinguishable elements

- **Restrictions** on f
  - f is arbitrary (no restriction).
  - f is injective (one-to-one). Need $k \leq n$.
  - f is surjective (onto). Need $k \geq n$.
- Whether the elements of X and S are **distinguishable** or **indistinguishable**.

Case 1 vs. Case 2

- Case 1: n^k
  - k-element ordered objects sampled with replacement
  - arbitrary $f: X \to S$; both X and S contain distinguishable elements
- Case 2: $n(n-1)\cdots(n-k+1)$
  - k-element ordered objects sampled without replacement
  - injective $f: X \to \text{S}$; both X and S contain distinguishable elements

For any fixed k: $$\lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} = 1$$

When the sample size k is small compared to the population size n, sampling with and without replacement are similar. Most of the functions are injective for $k \ll n$.

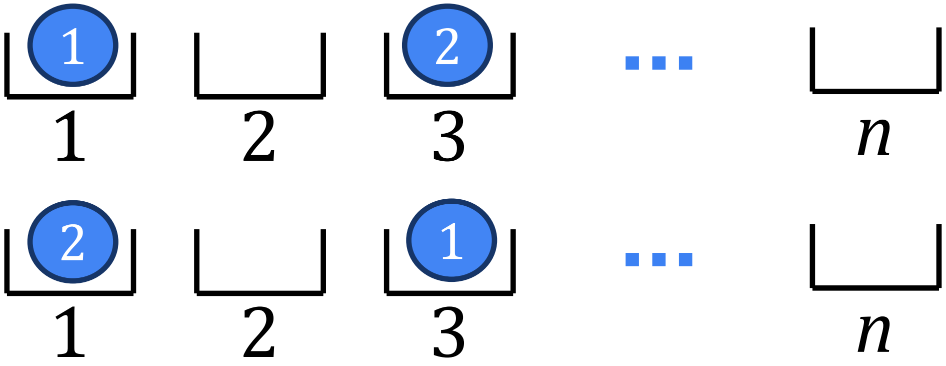
Stirling's approximation of $n!$

For $k = n$: $$\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n^n} = \lim_{n\to\infty} \sqrt{2\pi n}\frac{1}{e^n} = 0$$

Almost no functions are injective for large $k = n$.

Balls and Bins

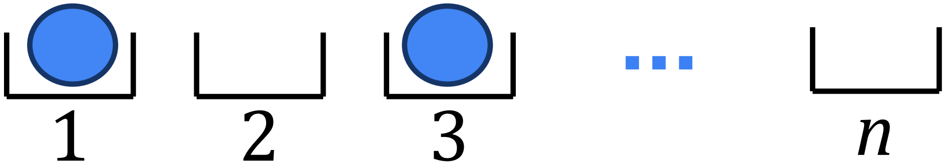
$$f: X \to S, \text{ where } |X| = k \text{ and } |S| = n$$



Elements (balls) of X are **distinguishable**
Elements (bins) of S are distinguishable

These two configurations are **inequivalent**

The above two configurations become **equivalent** if the labels on the balls get removed.



Elements (balls) of X are **indistinguishable**
Elements (bins) of S are distinguishable

Case 3: Unordered objects sampled without replacement

Injective $f: X \to S$; the elements (balls) in X are indistinguishable while the elements (bins) in S are distinguishable

- For every configuration with **unlabeled** balls, there correspond $\boldsymbol{k!}$ configurations with **labeled** balls.
- Hence, the number of injections $f: X \to S$ where X contains indistinguishable elements while S contains distinguishable elements is given by

$$\frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = \binom{n}{n-k}$$

Second Rule of Counting

- Suppose an object is formed by a succession of k choices, but the order does not matter.

- If there exists an m-to-1 surjective map

$$g: \{\text{ordered objects}\} \to \{\text{unordered objects}\}$$

(i.e., for every $x \in \{\text{unordered objects}\}, |g^{-1}(x)| = m$), then

$$|\{\text{unordered objects}\}| = \frac{|\{\text{ordered objects}\}|}{m}$$

- Count as if order matters and then divide by m.

Second Rule of Counting

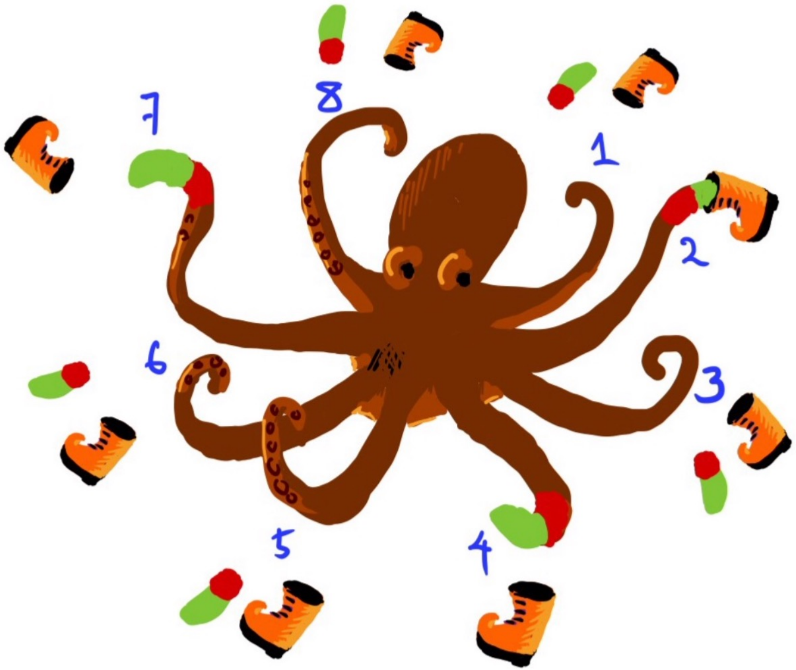
- More generally, if there exists an m-to-1 surjective map

$$g: A \to B$$

(i.e., for every $x \in B$, $|g^{-1}(x)| = m$), then

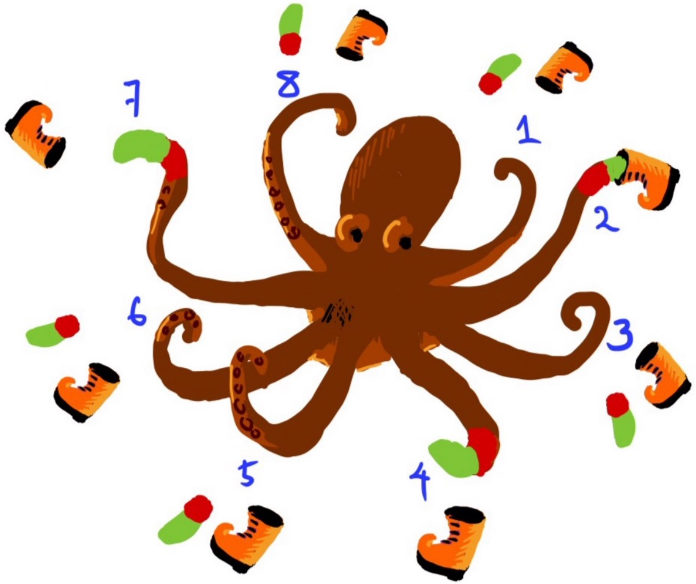
$$|B| = \frac{|A|}{m}$$

Example 1



- In how many different orders can an octopus put on its socks and shoes on its 8 limbs?
- For humans there are 6 ways:
  1. (Left sock, Left shoe, Right sock, Right shoe)
  2. (Left sock, Right sock, Left shoe, Right shoe)
  3. (Left sock, Right sock, Right shoe, Left shoe)
  4. (Right sock, Left sock, Left shoe, Right shoe)
  5. (Right sock, Left sock, Right shoe, Left shoe)
  6. (Right sock, Right shoe, Left sock, Left shoe)

Example 1

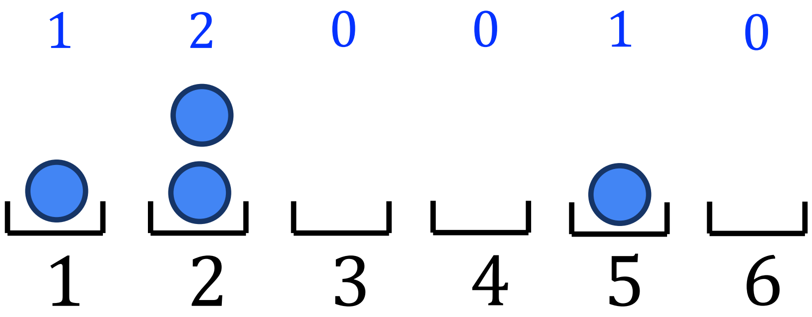


Socks: $X_1, X_2, \ldots, X_8$
Shoes: $Z_1, Z_2, \ldots, Z_8$

1. # Ways to order these 16 items = **16!** (by the First Rule of Counting). Let A denote the set of these length-16 sequences
2. But, not every $s \in A$ satisfies the constraint that **sock X_i should appear before shoe Z_i**. Let B denote the subset of **valid sequences** in A.
3. Each $s \in A$ can be mapped (g) to a valid sequence in B by **swapping Z_i with X_i if Z_i appears before X_i**.
4. $g: A \rightarrow B$ is a many-to-one surjection. For every $v \in B, |g^{-1}(v)| = \mathbf{2^8}$.
5. Hence, by the Second Rule of Counting, we obtain $\frac{\mathbf{16!}}{\mathbf{2^8}}$ for the answer.

Case 4: Unordered objects sampled with replacement

Arbitrary $f: X \rightarrow S$; the elements (balls) in X are indistinguishable while the elements (bins) in S are distinguishable.

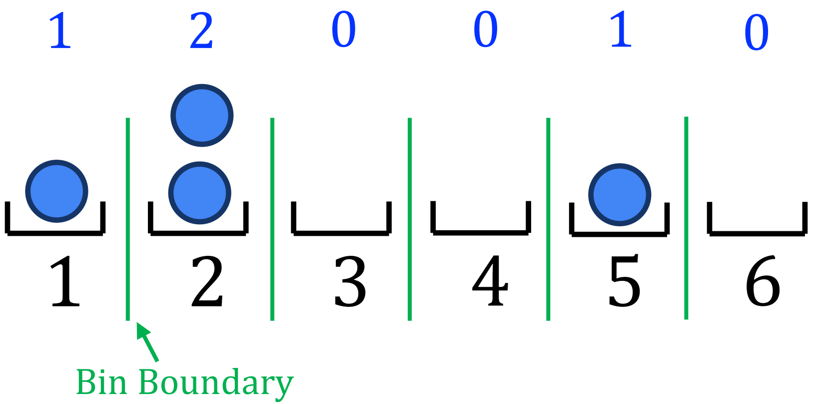


← Number of balls in each bin

4 **unlabeled** balls into
6 **labeled** bins,
no restriction on occupancy

Case 4: Unordered objects sampled with replacement

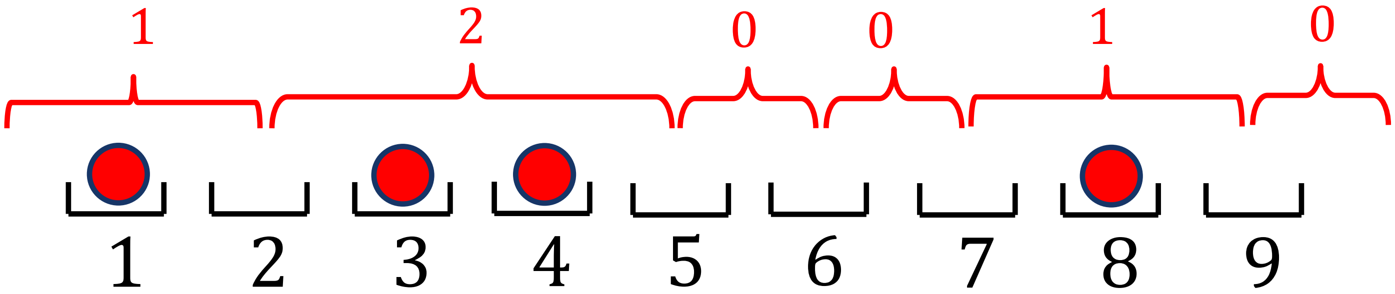
Arbitrary $f: X \to S$; the elements (balls) in X are indistinguishable while the elements (bins) in S are distinguishable.



Create a new configuration with k balls and $n+k-1$ bins following this procedure, moving from left to right:

- For each **ball**, assign (a bin holding one ball) (k balls in total)
- For each **bin boundary**, assign (an empty bin) ($n-1$ bin boundaries in total)
- Label the new bins $1, \ldots, n+k-1$

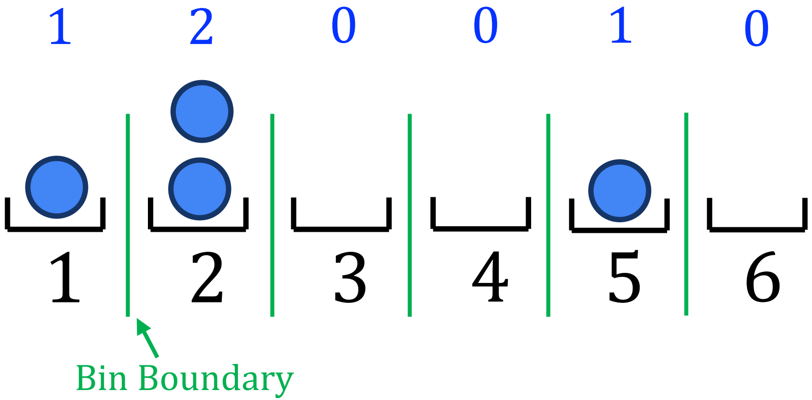
The number of consecutive occupied bins between empty bins, treating the two ends as imaginary empty bins



Note that these numbers correspond to the occupancy numbers in the original configuration above.

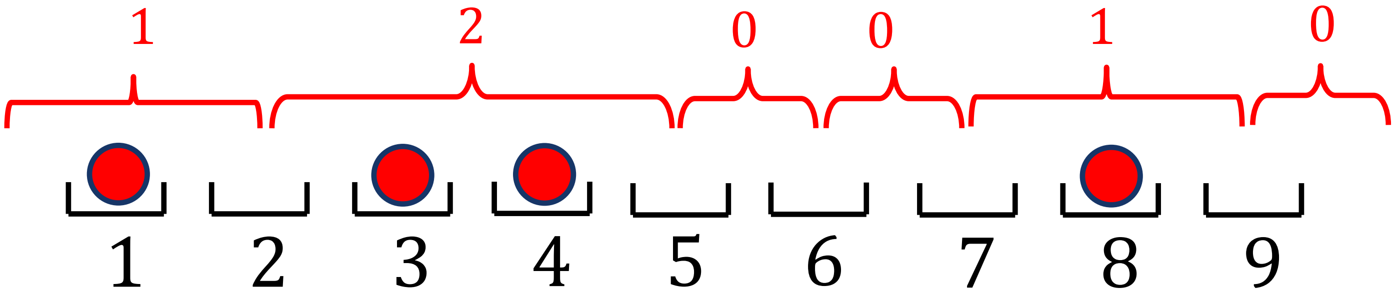
Case 4: Unordered objects sampled with replacement

Arbitrary $f: X \to S$; the elements (balls) in X are indistinguishable while the elements (bins) in S are distinguishable.



Conversely, given any configuration with k balls and $n + k - 1$ labeled bins such that each bin contains at most one ball, the previous procedure can be **reversed** to create a **unique** configuration with k balls and n labeled bins where each bin can have multiple balls.

The number of consecutive occupied bins between empty bins, treating the two ends as imaginary empty bins



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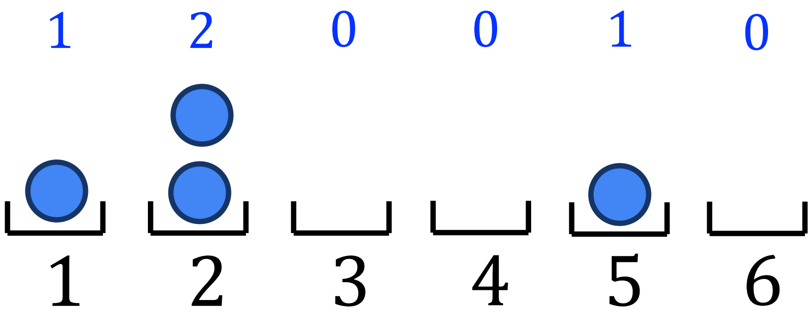
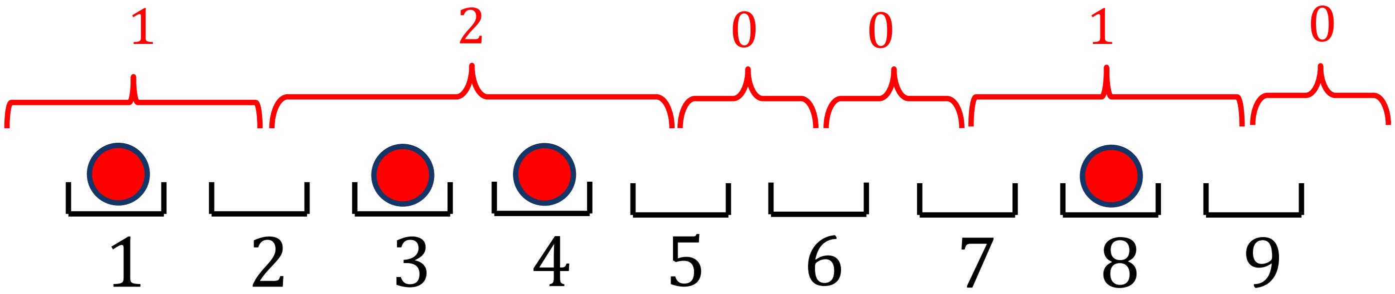


k unlabeled balls into n labeled bins, no restrictions

Bijection

k unlabeled balls into $n + k - 1$ labeled bins, with ≤ 1 ball per bin

The number of consecutive occupied bins between empty bins, treating the two ends as imaginary empty bins



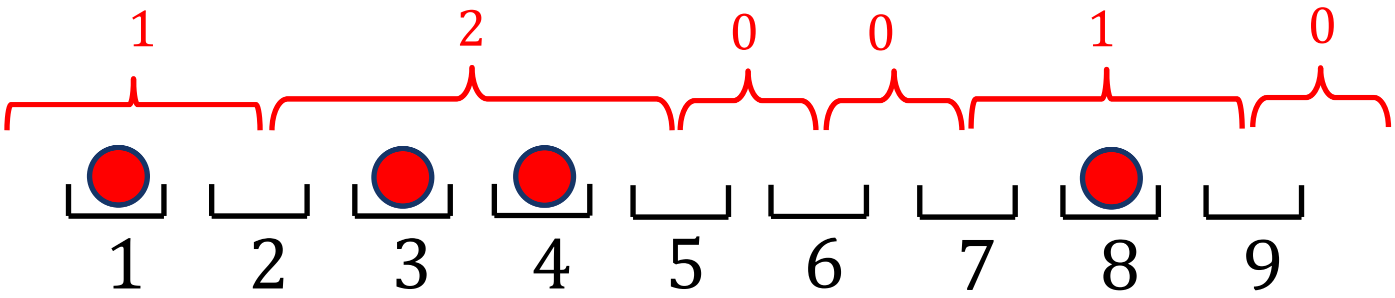
Note that these numbers correspond to the occupancy numbers in the original configuration above.

Zeroth Rule of Counting

- If there exists a bijection (one-to-one correspondence) between two finite sets A and B, then $|A| = |B|$.

Case 4: Unordered objects sampled with replacement

Arbitrary $f : X \to S$; the elements (balls) in X are indistinguishable while the elements (bins) in S are distinguishable



Sometimes this representation is referred to "**stars and bars**", where "stars" correspond to balls and "bars" to empty bins.

From **Case 3**, we know how to count the number of configurations for k unlabeled balls and $n + k - 1$ labeled bins, with **at most one ball per bin**:

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

Example 2

Let n and k be positive integers. How many solutions does the following equation admit?

$$x_1 + x_2 + \cdots + x_n = k, \text{ where } x_i \in \mathbb{N}.$$

Answer: $\binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ *k unlabeled balls into n labeled bins, no restrictions*

Let n and $k > n$ be positive integers. How many solutions does the following equation admit?

$$x_1 + x_2 + \cdots + x_n = k, \text{ where } x_i \in \mathbb{N} \setminus \{0\}.$$

$$x_1 + x_2 + \cdots + x_n = k - n, \text{ where } x_i \in \mathbb{N}.$$

Answer: $\binom{k-1}{k-n} = \binom{k-1}{n-1}$ *There exists a bijection between the solutions to these two equations.*

Summary

$$f: X \to S, \text{ where } |X| = k \text{ and } |S| = n$$

	Elements of X	Elements of S	Arbitrary f (Sampling with replacement)	Injective f (Sampling without replacement)
Ordered	Distinguishable	Distinguishable	Case 1 n^k	Case 2 $\frac{n!}{(n-k)!}$
Unordered	Indistinguishable	Distinguishable	Case 4 $\binom{n+k-1}{k}$	Case 3 $\binom{n}{k}$