1 Proof Practice

(a) Prove that $\forall n \in \mathbb{N}$, if $n$ is odd, then $n^2 + 1$ is even. (Recall that $n$ is odd if $n = 2k + 1$ for some natural number $k$.)

(b) Prove that $\forall x, y \in \mathbb{R}$, $\min(x, y) = (x + y - |x - y|)/2$. (Recall, that the definition of absolute value for a real number $z$, is $|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$)

(c) Suppose $A \subseteq B$. Prove $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. (Recall that $A' \in \mathcal{P}(A)$ if and only if $A' \subseteq A$.)

Solution:

(a) We will use a direct proof. Assume $n$ is odd. By the definition of odd numbers, $n = 2k + 1$ for some natural number $k$. Substituting into the expression $n^2 + 1$, we get $(2k + 1)^2 + 1$. Simplifying the expression yields $4k^2 + 4k + 2$. This can be rewritten as $2 \times (2k^2 + 2k + 1)$. Since $2k^2 + 2k + 1$ is a natural number, by the definition of even numbers, $n^2 + 1$ is even.

(b) We will use a proof by cases. Again, the definition of the absolute value function for real number $z$ is $|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$

Case 1: $x < y$. This means $|x - y| = y - x$. Substituting this into the formula on the right hand side, we get

$$\frac{x + y - y + x}{2} = x = \min(x, y).$$

Case 2: $x \geq y$. This means $|x - y| = x - y$. Substituting this into the formula on the right hand side, we get

$$\frac{x + y - x + y}{2} = y = \min(x, y).$$

(c) Suppose $A' \in \mathcal{P}(A)$, that is, $A' \subseteq A$ (by the definition of the power set). We must prove that for any such $A'$, we also have that $A' \in \mathcal{P}(B)$, that is, $A' \subseteq B$.

Let $x \in A'$. Then, since $A' \subseteq A$, $x \in A$. Since $A \subseteq B$, $x \in B$. We have shown $\forall x \in A'$ $x \in B$, so $A' \subseteq B$.

Since the previous argument works for any $A' \subseteq A$, we have proven $\forall A' \in \mathcal{P}(A)) A' \in \mathcal{P}(B)$. So, we conclude $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ as desired.
2 Preserving Set Operations

For a function \( f \), define the image of a set \( X \) to be the set \( f(X) = \{ y \mid y = f(x) \text{ for some } x \in X \} \). Define the inverse image or preimage of a set \( Y \) to be the set \( f^{-1}(Y) = \{ x \mid f(x) \in Y \} \). Prove the following statements, in which \( A \) and \( B \) are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets \( X \) and \( Y \), \( X = Y \) if and only if \( X \subseteq Y \) and \( Y \subseteq X \). To prove that \( X \subseteq Y \), it is sufficient to show that \((\forall x) \ ((x \in X) \implies (x \in Y))\).

(a) \( f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \).
(b) \( f(A \cup B) = f(A) \cup f(B) \).

Solution:

In order to prove equality \( A = B \), we need to prove that \( A \) is a subset of \( B \), \( A \subseteq B \) and that \( B \) is a subset of \( A \), \( B \subseteq A \). To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose \( x \in f^{-1}(A \cup B) \) which means that \( f(x) \in A \cup B \). Then either \( f(x) \in A \), in which case \( x \in f^{-1}(A) \), or \( f(x) \in B \), in which case \( x \in f^{-1}(B) \), so in either case we have \( x \in f^{-1}(A) \cup f^{-1}(B) \). This proves that \( f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B) \).

Now, suppose that \( x \in f^{-1}(A) \cup f^{-1}(B) \). Suppose, without loss of generality, that \( x \in f^{-1}(A) \). Then \( f(x) \in A \), so \( f(x) \in A \cup B \), so \( x \in f^{-1}(A \cup B) \). The argument for \( x \in f^{-1}(B) \) is the same.

Hence, \( f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B) \).

(b) Suppose that \( x \in A \cup B \). Then either \( x \in A \), in which case \( f(x) \in f(A) \), or \( x \in B \), in which case \( f(x) \in f(B) \). In either case, \( f(x) \in f(A) \cup f(B) \), so \( f(A \cup B) \subseteq f(A) \cup f(B) \).

Now, suppose that \( y \in f(A) \cup f(B) \). Then either \( y \in f(A) \) or \( y \in f(B) \). In the first case, there is an element \( x \in A \) with \( f(x) = y \); in the second case, there is an element \( x \in B \) with \( f(x) = y \). In either case, there is an element \( x \in A \cup B \) with \( f(x) = y \), which means that \( y \in f(A \cup B) \). So \( f(A) \cup f(B) \subseteq f(A \cup B) \).

The purpose of this problem is to gain familiarity to naming thing precisely. In particular, we named an element in the LHS (or the pre-image of the LHS) and then argued about whether that element or its image was in the right hand side. By explicitly naming an element generically where it could be any element in the set, we could argue about its membership in a set and or its image or preimage. With these different concepts floating around it is helpful to be clear in the argument.

3 Fermat’s Contradiction

Prove that \( 2^{1/n} \) is not rational for any integer \( n \geq 3 \). (Hint: Use Fermat’s Last Theorem. It states that there exists no positive integers \( a, b, c \) s.t. \( a^n + b^n = c^n \) for \( n \geq 3 \).)
Solution:
If not, then there exists an integer \( n \geq 3 \) such that \( 2^{1/n} = \frac{p}{q} \) where \( p, q \) are positive integers. Thus, \( 2q^n = p^n \), and this implies
\[
q^n + q^n = p^n,
\]
which is a contradiction to the Fermat’s Last Theorem.

4 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.

**Solution:** We give a proof by contraposition. Suppose there does not exist an all-red column. This means that, in each column, we can find a blue pebble. Therefore, if we take one blue pebble from each column, we have a way of choosing one pebble from each column without any red pebbles. This is the negation of the original hypothesis, so we are done.