1 Proof Practice

(a) Prove that \( \forall n \in \mathbb{N}, \) if \( n \) is odd, then \( n^2 + 1 \) is even. (Recall that \( n \) is odd if \( n = 2k + 1 \) for some natural number \( k \).)

(b) Prove that \( \forall x, y \in \mathbb{R}, \min(x, y) = (x + y - |x - y|)/2. \) (Recall, that the definition of absolute value for a real number \( z \), is 
\[
|z| = \begin{cases} 
  z, & z \geq 0 \\
  -z, & z < 0 
\end{cases}
\]

(c) Suppose \( A \subseteq B. \) Prove \( \mathcal{P}(A) \subseteq \mathcal{P}(B). \) (Recall that \( A' \in \mathcal{P}(A) \) if and only if \( A' \subseteq A \).)

2 Preserving Set Operations

For a function \( f \), define the image of a set \( X \) to be the set \( f(X) = \{ y \mid y = f(x) \text{ for some } x \in X \} \). Define the inverse image or preimage of a set \( Y \) to be the set \( f^{-1}(Y) = \{ x \mid f(x) \in Y \} \). Prove the following statements, in which \( A \) and \( B \) are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets \( X \) and \( Y, X = Y \) if and only if \( X \subseteq Y \) and \( Y \subseteq X \). To prove that \( X \subseteq Y \), it is sufficient to show that \( (\forall x) ((x \in X) \implies (x \in Y)). \)

(a) \( f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B). \)
(b) \( f(A \cup B) = f(A) \cup f(B) \).

3 Fermat’s Contradiction

Prove that \( 2^{1/n} \) is not rational for any integer \( n \geq 3 \). (Hint: Use Fermat’s Last Theorem. It states that there exists no positive integers \( a, b, c \) s.t. \( a^n + b^n = c^n \) for \( n \geq 3 \).)

4 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.