CS 70 Discrete Mathematics and Probability Theory
Spring 2018 Satish Rao and Babak Ayazifar **DIS 1B**

1 Stable Marriage

Consider the set of men $M = \{1,2,3\}$ and the set of women $W = \{A,B,C\}$ with the following preferences.

Men	Women		
1	A	B	C
2	B	A	C
3	A	B	C

Women	Men		
A	2	1	3
B	1	2	3
C	1	2	3

Run the male propose-and-reject algorithm on this example. How many days does it take and what is the resulting pairing? (Show your work.)

Solution:

The algorithm takes 3 days to produce a matching. The resulting pairing is as follows. The circles indicate the man that a woman picked on a given day (and rejected the rest).

$$\{(A,1),(B,2),(C,3)\}.$$

Woman	Day 1	Day 2	Day 3
A	1,3	1	1
B	2	2,3	2
C			3

2 Stable Marriage

The following questions refer to stable marriage instances with n men and n women, answer True/-False or provide an expression as requested.

(a) For $n = 2$, or any 2-man, 2-woman stable marriage instance, man A has the same optimal and pessimal woman. (True or False.)

(b) In any stable marriage instance, in the pairing the TMA (traditional stable marriage algorithm) produces there is some man who gets his favorite woman (the first woman on his preference list). (True or False.)

(c) In any stable marriage instance with n men and women, if every man has a different favorite woman, a different second favorite, a different third favorite, and so on, and every woman has the same preference list, how many days does it take for TMA to finish? (Form of Answer: An expression that may contain n.)

(d) Consider a stable marriage instance with n men and n women, and where all men have the same preference list, and all women have different favorite men, and different second-favorite men, and so on. How many days does the TMA take to finish? (Form of Answer: An expression that may contain n.)

(e) It is possible for a stable pairing to have a man A and a woman 1 be paired if A is 1's least preferred choice and 1 is A's least preferred choice. (True or False.)

(f) It is possible for a stable pairing to have two couples where each person is paired with their least favorite choice. (True or False.)

(g) If there is a pairing, P, that consists of only pairs from male and female optimal pairings, then it must be stable. In other words, if every pair in P is a pair either in the male-optimal or the female-optimal pairing then P is stable. (True or False.)

Solution:

(a) **False.** This says there is only one stable pairing. But if the preference list for man A is $(1,2)$ and for man B is $(2,1)$ and preference list for woman 1 is (B,A) and woman 2 is (A,B) produce different male and female optimal pairings.

(b) **False.** Let man A have preference list $(1,3,2)$, B have $(1,2,3)$, and C have $(2,1,3)$. We develop a "cyclic" chain of preferences, causing A to displace B to displace C who then displaces A.

   (a) If woman 1 prefers A over B, she puts A on a string and rejects B.
   (b) B does not get his favorite and proposes to 2, who prefers B over C and thus rejects C.
   (c) C does not get his favorite and proposes to 1, who prefers C over A and thus rejects A.

   Thus, A also does not get his favorite, and no man gets his favorite.

(c) **1**.
On the first day every woman gets a proposal since each man has a different woman in their first position. The algorithm terminates.

(d) **n**.
Every man proposes to their common favorite. One man is kept on the string. The rest propose to the second. And so one. After each day, a new woman gets a man on a string. After n days, we finish. Note that the men's preference lists (assuming they're the same for everyone) were irrelevant.

(e) **True.**
A and 1 are respectively all the women's and men's least favorite.

(f) **False.**

Assume for the sake of contradiction that it is possible to have two couples in a pairing, where each person is paired with their least favorite choice. Consider (m,w) and (m^*,w^*), where m and w are paired despite being each others' last choices, as are m^* and w^*. Since m is w's last choice, she prefers m^* to her current partner. But w^* is m^*'s last choice, so he also prefers w to his current partner. Thus, w and m^* form a rogue couple, showing that no pairing with this situation can be stable.

(g) **False**.

Consider a woman who is matched to her pessimal partner and a man who is matched to his pessimal partner. They may well like each other.

An example is as follows.

Men's preference list
A: $1 > \ldots > 2$
B: $2 > \ldots > 1$
C: $3 > \ldots > 4$
D: $4 > \ldots > 3$

Women's preference list
1: $B > \ldots > A$
2: $A > \ldots > B$
3: $D > \ldots > C$
4: $C > \ldots > D$

Men's first choices = women's last choices and vice versa.

men-optimal: $(A,1)$, $(B,2)$, $(C,3)$, $(D,4)$
women-optimal: $(B,1)$, $(A,2)$, $(D,3)$, $(C,4)$
our pairing: $(A,1)$, $(B,2)$, $(D,3)$, $(C,4)$. Note that $1,2,C$, and D are all with their pessimal partner, so any pairing of 1 or 2 with C or D will be rogue. For example, $(C,1)$ is a rogue couple.

3 Good, Better, Best

In a particular instance of the stable marriage problem with n men and n women, it turns out that there are exactly three distinct stable matchings, S_1, S_2, and S_3. Also, each man m has a different partner in the three matchings. Therefore each man has a clear preference ordering of the three matchings (according to the ranking of his partners in his preference list). Now, suppose for man m_1, this order is $S_1 > S_2 > S_3$.

Prove that every man has the same preference ordering $S_1 > S_2 > S_3$.

Hint: Recall that a male-optimal matching always exists and can be generated using TMA. By reversing the roles of TMA, what other matching can we generate?

Solution:

In class, you were given the traditional propose-and-reject algorithm, which was guaranteed to produce a male-optimal matching. By switching men's and women's roles, you would be guaranteed to produce a female-optimal matching, which, by a lemma from class, would also be male-pessimal. By the very fact that these algorithms exist and have been proven to work in this way, you're guaranteed that a male-optimal and a male-pessimal matching always exist.

Since there are only three matchings in this particular stable matching instance, we thus know that one of them must be male-optimal and one must be male-pessimal. Since m_1 prefers S_1 above the other stable matchings, only that one can be male-optimal by definition of male-optimality. Similarly, since m_1 prefers S_3 the least, it must be the male-pessimal. Therefore, again from definitions of optimality/pessimality, since each men have different matches in the three stable matchings, they *must* strictly prefer S_1 to both of the others, and they *must* like S_3 strictly less than both of the others. Thus, each man's preference order of stable matchings must be S_1, S_2, S_3.