CS 70 Summer 2017 | Discrete Mathematics and Probability Theory Lu, Moulos, and Tang | DIS 1B

1 Short Proofs

(a) Prove that $\sqrt[12]{2}$ is irrational.

(b) Prove that $\sum_{i=1}^{n} i = n(n+1)/2$.

Solution:

(a) Proof by contradiction. Suppose $\sqrt[12]{2}$ was rational. We know that for any rational value, x, x^6 must also be rational. If $x = \sqrt[12]{2}$ and x is rational, then $x^6 = \sqrt{2}$ must also be rational. However, we know that $\sqrt{2}$ is in fact irrational. We have a contradiction and therefore $\sqrt[12]{2}$ must be irrational.

(b)

$$\sum_{i=1}^{n} i = 1+2+\cdots+n$$

$$2\sum_{i=1}^{n} i = (1+n)+(2+(n-1))+\cdots+(n+1) = (n+1)n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

2 Infinite Primes

Prove by contradiction that there are an infinite number of primes.

Solution:

We assume there are a finite number n of primes, $p_1,\ldots,p_n$. Let $m = p_1 \cdots p_n + 1$. We know m is either prime or divisible by a prime; m is not divisible by a prime by construction, since we will have remainder 1. Clearly, $m > p_n$, so m can not be prime because p_n is the largest prime. Thus we have a contradiction, and there must be an infinite number of primes.

3 Proof by?

(a) Prove that if $x, y \in \mathbb{Z}$, if 10 does not divide xy, then 10 does not divide x and 10 does not divide y. In notation: $(\forall x, y \in \mathbb{Z})\ \ 10 \nmid xy \implies (10 \nmid x \ \wedge \ 10 \nmid y)$. What proof technique did you use?

(b) Prove or disprove the contrapositive.

(c) Prove or disprove the converse.

Solution:

(a) We will use proof by contraposition. For any arbitrary given x and y, the statement $10 \nmid xy \implies (10 \nmid x \wedge 10 \nmid y)$ is equivalent using contraposition to $\neg(10 \nmid x \wedge 10 \nmid y) \implies \neg(10 \nmid xy)$. Moving the negations inside, this becomes equivalent to $(10 \mid x \vee 10 \mid y) \implies 10 \mid xy$.

Now for this part, we give a proof by cases. Assuming that $10 \mid x \vee 10 \mid y$, one of the two cases must be true.

  (a) $10 \mid x$: in this case $x = 10k$ for some $k \in \mathbb{Z}$. Therefore $xy = 10ky$ which is a multiple of 10. So $10 \mid xy$.

  (b) $10 \mid y$: in this case $y = 10k$ for some $k \in \mathbb{Z}$. Therefore $xy = 10kx$ which is a multiple of 10. So $10 \mid xy$.

Therefore assuming $10 \mid x \vee 10 \mid y$ we proved $10 \mid xy$.

We used proof by cases and proof by contraposition.

(b) We proved the statement. The contrapositive of a statement has logically equivalent to the statement. So we are done.

(c) Its not true! The converse is that if 10 does not divide x and does not divide y then 10 does not divide xy. We can choose $x = 2$ and $y = 5$ and see a counterexample to the statement.

4 Perfect Square

A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

Solution:

Let $n = m^2$ for some integer m. Since n is odd, m is also odd, i.e., of the form $m = 2l+1$ for some integer l. Then, $m^2 = 4l^2 + 4l + 1 = 4l(l+1) + 1$. Since one of l and $l+1$ must be even, $l(l+1)$ is of the form $2k$ and $n = m^2 = 8k+1$.