CS 70 Discrete Mathematics and Probability Theory
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DIS 1B

1 Triangle Inequality

Recall the triangle inequality, which states that for real numbers x_1 and x_2,

$$|x_1 + x_2| \leq |x_1| + |x_2|.$$

Assuming the above inequality holds, use induction to prove the generalized triangle inequality:

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Solution:

We use induction on $n \geq 2$. The base case $n = 2$ is the usual triangle inequality. Assume the inequality holds for some $n \geq 2$ (this is the inductive hypothesis). Mathematically, that is:

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

For $n+1$, we want to prove:

$$|x_1 + x_2 + \cdots + x_n + x_{n+1}| \leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|$$

Applying the original triangle inequality to the real numbers $x_1 + x_2 + \cdots + x_n$ and x_{n+1}, we get:

$$|(x_1 + x_2 + \cdots + x_n) + x_{n+1}| \leq |x_1 + x_2 + \cdots + x_n| + |x_{n+1}|$$

Adding $|x_{n+1}|$ to both sides of the inequality from the **induction hypothesis**, we get:

$$|x_1 + x_2 + \cdots + x_n| + |x_{n+1}| \leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|$$

Combining these two inequalities, we have:

$$\therefore |x_1 + x_2 + \cdots + x_n + x_{n+1}| \leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|$$

This completes the induction.

2 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{2^n}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply $a_n \leq 3^{2^n}$? Show why this does not work.

(b) Try to instead prove the statement $a_n \leq 3^{2^n - 1}$ using induction. Does this statement imply what you tried to prove in the previous part?

Solution:

(a) Try to prove that for every $n \geq 1$, we have $a_n \leq 3^{2^n}$ by induction.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2^1} = 9$.
Inductive Hypothesis: For $n \geq 1$ we assume $a_n \leq 3^{2^n}$.
Inductive Step: Assuming the statement is true for an n, we have

$$a_{n+1} = 3a_n^2 \leq 3(3^{2^n})^2 = 3 \times 3^{2 \times 2^n} = 3 \times 3^{2^{n+1}} = 3^{2^{n+1}+1}.$$

However, what we wanted was to get an inequality of the form: $a_{n+1} \leq 3^{2^{n+1}}$. There is an extra $+1$ in the exponent of what we derived.

(b) This time the induction works.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2-1} = 3$.
Inductive Hypothesis: For $n \geq 1$ we assume $a_n \leq 3^{2^n - 1}$.
Inductive Step: Assuming the hypothesis holds for n, we get

$$a_{n+1} = 3a_n^2 \leq 3 \times (3^{2^n - 1})^2 = 3 \times 3^{2 \times (2^n - 1)} = 3 \times 3^{2^{n+1} - 2} = 3^{2^{n+1} - 1}.$$

This is exactly the induction hypothesis for $n+1$. Note that for every $n \geq 1$, we have $2^n - 1 \leq 2^n$ and therefore $3^{2^n - 1} \leq 3^{2^n}$. This means that our modified hypothesis which we proved here does indeed imply what we wanted to prove in the previous part. This is called "strengthening" the induction hypothesis because we proved a stronger statement and by proving that statement to be true, we proved our original statement to be true as well.

3 Bit String

Prove that every positive integer n can be written with a string of 0s and 1s. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0,1\}$.

Solution:

Prove by strong induction on n. Note that this is the first time students will have seen strong induction, so it is important that this problem be done in an interactive way that shows them how simple induction gets stuck.

- *Base Case:* $n = 1$ can be written with 1×2^0.
- *Inductive Hypothesis:* Assume that the statement is true for all $1 \leq k \leq n$.
- *Inductive Step:* If $n+1$ is divisible by 2, then it can use the representation of $(n+1)/2$.

$$\begin{aligned}(n+1)/2 &= c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0 \\ n+1 = 2 \cdot (n+1)/2 &= c_k \cdot 2^{k+1} + c_{k-1} \cdot 2^k + \cdots + c_1 \cdot 2^2 + c_0 \cdot 2^1.\end{aligned}$$

  Otherwise, n must be divisible by 2 and have $c_0 = 0$. We can obtain the representation of $n+1$ from n.

$$\begin{aligned}n &= c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + 0 \cdot 2^0 \\ n+1 &= c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + 1 \cdot 2^0\end{aligned}$$

  Therefore, the statement is true.

Note: In proofs using simple induction, we only use $P(n)$ in order to prove $P(n+1)$. Simple induction gets stuck here because in order to prove $P(n+1)$ in the inductive step, we need to assume more than just $P(n)$. This is because it is not immediately clear how to get a representation for $P(n+1)$ using just $P(n)$. As a result, we assume the statement to be true for all of $1, 2, \ldots, n$ in order to prove it for $P(n+1)$.

4 Well-Ordering Principle

In this question, we will go over how the well-ordering principle can be derived from (strong) induction. Remember the well-ordering principle states the following:

For every non-empty subset S of the set of natural numbers $\mathbb{N}$, there is a smallest element $x \in S$; i.e.

$$\exists x : \forall y \in S : x \leq y.$$

(a) What is the significance of S being non-empty? Does WOP hold without it? Assuming that S is not empty is equivalent to saying that there exists some number z in it.

(b) Induction is always stated in terms of a property that can only be based on a natural number. What should the induction be based on? The length of the set S? The number x? The number y? The number z?

(c) Now that the induction variable is clear, state the induction hypothesis. Be very precise. Do not leave out dangling symbols other than the induction variable. Ideally you should be able to write this in mathematical notation.

(d) Verify the base case. Note that your base case does not just consist of a single set S.

(e) Now prove that the induction works, by writing the inductive step.

(f) What should you change so that the proof works by simple induction (as opposed to strong induction)?

Solution:

(a) If S is empty, then WOP does not hold obviously. The significance is that with S not empty, you can always take a number out of it and start from there.

(b) The induction is based on z—an arbitrary natural number. By induction, we aim to prove our statement for all such z.

(c) **Hypothesis:** For all sets S that contain an element s, where $s \leq z$, the set S contains a smallest element.
In pure mathematical notation, this is: $\forall S \subseteq \mathbb{N}, (\exists s \in S,\ s \leq z) \implies \exists x : \forall y \in S : x \leq y$

(d) **Base Case:** For $z = 0$, the claim is true. Because we can take $x = 0$ as the smallest element and for all $y \in S$ we have $y \in \mathbb{N}$, and therefore $y \geq 0 = x$.

(e) **Inductive Step:** Let S be a set that contains z. If z is the smallest element, we are done. Otherwise there exists $y \in S$, such that $y < z$. But now, by the induction hypothesis for y, we know that S contains a smallest element.

(f) In order to make the proof work without strong induction, one can modify the induction hypothesis in the following way: **Hypothesis (in terms of z)** For all sets S that contain a number z' such that $z' \leq z$, the set S contains a smallest element.

This makes the proof work, simply because when the set contains a smaller element than z, we know that the smaller element is less than or equal to $z - 1$, which allows us to appeal to the inductive hypothesis for $z - 1$. Otherwise, if there are no smaller elements than z, z must be the smallest element.