1 Logic

Decide whether each of the following is true or false and justify your answer:

(a) $\forall x \, \big(P(x) \land Q(x)\big) \; \equiv \; \forall x \, P(x) \land \forall x \, Q(x)$

(b) $\forall x \, \big(P(x) \lor Q(x)\big) \; \equiv \; \forall x \, P(x) \lor \forall x \, Q(x)$

(c) $\exists x \, \big(P(x) \lor Q(x)\big) \; \equiv \; \exists x \, P(x) \lor \exists x \, Q(x)$

(d) $\exists x \, \big(P(x) \land Q(x)\big) \; \equiv \; \exists x \, P(x) \land \exists x \, Q(x)$

Solution:

(a) **True**.

We start by assuming the LHS. Then we know for an arbitrary x $P(x) \land Q(x)$ is true. This means that both $\forall x P(x)$ and $\forall x Q(x)$. The RHS follows. Now assume the RHS. Since for any x $P(x)$ and for any y $Q(y)$ holds, then for an arbitrary x both $P(x)$ and $Q(x)$ must be true. The LHS follows.

(b) **False**. If $P(1)$ is true, $Q(1)$ is false, $P(2)$ is false and $Q(2)$ is true, the left-hand side will be true, but the right-hand side will be false.

(c) **True**

Assuming the LHS, we know there exists some x such that one of $P(x)$ and $Q(x)$ is true. Thus $\exists x P(x)$ or $\exists x Q(x)$ and the RHS holds. To prove the other direction, assume the LHS is false. Then there does not exists an x for which $P(x) \lor Q(x)$ is true, which means there is no x for which $P(x)$ or $Q(x)$ is true. Therefore the RHS is false.

(d) **False**. If $P(1)$ is true and $P(x)$ is false for all other x, and $Q(2)$ is true and $Q(x)$ is false for all other x, the right hand side would be true. However, there would be no value of x at which both $P(x)$ and $Q(x)$ would be simultaneously true.

2 Contraposition

Prove the statement "if $a+b < c+d$, then $a < c$ or $b < d$".

Solution:

The implication we're trying to prove is $(a+b<c+d) \implies ((a<c) \vee (b<d))$, so the contrapositive is $((a \geq c) \wedge (b \geq d)) \implies (a+b \geq c+d)$. The proof of this is quite straightforward: since we have both that $a \geq c$ and that $b \geq d$, we can just add these two inequalities together, giving us $a+b \geq c+d$, which is exactly what we wanted.

3 Perfect Square

A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

Solution:

We will proceed with a direct proof. Let $n = m^2$ for some integer m. Since n is odd, m is also odd, i.e., of the form $m = 2l+1$ for some integer l. Then, $m^2 = 4l^2+4l+1 = 4l(l+1)+1$. Since one of l and $l+1$ must be even, $l(l+1)$ is of the form $2k$ and $n = m^2 = 8k+1$.

4 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \ldots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.
Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to n possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0, 1, ..., n-1$. The objects assigned to these containers are the people at the party. However, containers 0, $n-1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning n people to at most $n-1$ containers, and by the pigeonhole principle, at least one of the $n-1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.

5 Fermat's Contradiction

Prove that $2^{1/n}$ is not rational for any integer $n \geq 3$. (*Hint*: Use Fermat's Last Theorem. It states that there exists no positive integers a, b, c s.t. $a^n + b^n = c^n$ for $n \geq 3$.)

Solution:

If not, then there exists an integer $n \geq 3$ such that $2^{1/n} = \frac{p}{q}$ where p, q are positive integers. Thus, $2q^n = p^n$, and this implies

$$q^n + q^n = p^n,$$

which is a contradiction to the Fermat's Last Theorem.