1 Prove or Disprove

For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) \((\forall n \in \mathbb{N}) \text{ if } n \text{ is odd then } n^2 + 4n \text{ is odd.}\)

(b) \((\forall a, b \in \mathbb{R}) \text{ if } a + b \leq 15 \text{ then } a \leq 11 \text{ or } b \leq 4.\)

(c) \((\forall r \in \mathbb{R}) \text{ if } r^2 \text{ is irrational, then } r \text{ is irrational.}\)

(d) \((\forall n \in \mathbb{Z}^+) 5n^3 > n!.\) (Note: \(\mathbb{Z}^+\) is the set of positive integers)

2 Pigeonhole Principle

Prove the following statement: If you put \(n + 1\) balls into \(n\) bins, however you want, then at least one bin must contain at least two balls. This is known as the pigeonhole principle.
3 Numbers of Friends

Prove that if there are \( n \geq 2 \) people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice. (Hint: The Pigeonhole Principle states that if \( n \) items are placed in \( m \) containers, where \( n > m \), at least one container must contain more than one item. You may use this without proof.)