1 Divisibility Induction

Prove that for all \( n \in \mathbb{N} \) with \( n \geq 1 \), the number \( n^3 - n \) is divisible by 3. (Hint: recall the binomial expansion \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\))

2 Make It Stronger

Let \( x \geq 1 \) be a real number. Use induction to prove that for all positive integers \( n \), all of the entries in the matrix

\[
\begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix}^n
\]

are \( \leq xn \). (Hint 1: Find a way to strengthen the inductive hypothesis! Hint 2: Try writing out the first few powers.)
3 Binary Numbers

Prove that every positive integer \( n \) can be written in binary. In other words, prove that we can write

\[
n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,
\]

where \( k \in \mathbb{N} \) and \( c_k \in \{0, 1\} \).