1 Badminton Ranking

A team of n ($n \geq 2$) badminton players held a tournament, where every person plays with every other person exactly once, and there are no ties. Prove by induction that after the tournament, we can arrange the n players in a sequence, so that every player in the sequence has won against the person immediately to the right of him.

Solution:

Denote the n players by P_1 through P_n. Define a sequence of arrangement that satisfies the given condition to be a *valid* sequence.

For $n = 2$, either P_1 wins against P_2 or P_2 wins against P_1, and a valid sequence is given by P_1P_2 in the first case, and P_2P_1 in the second case.

Assume that we can construct a valid sequence for $n = k$. For $n = k+1$, we can arrange the first k players in a valid sequence. Denote this sequence by $P_{i_1} \cdots P_{i_k}$, where $i_1 \cdots i_k$ is a permutation of 1 through k. Now we just need to insert P_{k+1} in the appropriate position. We proceed as follows:

1. If P_{k+1} wins against P_{i_1}, we can put P_{k+1} in the beginning of the sequence; otherwise we go to the next step.
2. If P_{k+1} wins against P_{i_2}, we can put P_{k+1} between P_{i_1} and P_{i_2}; otherwise we go to the next step.
3. ...

k. If P_{k+1} wins against P_{i_k}, we can put P_{k+1} between $P_{i_{k-1}}$ and P_{i_k}; otherwise we simply put P_{k+1} at the end of the sequence.

In all these situations, we can successfully construct a valid sequence of P_1 through P_{k+1}. Hence the induction step is proven, and the original claim follows.

2 Seating Arrangement

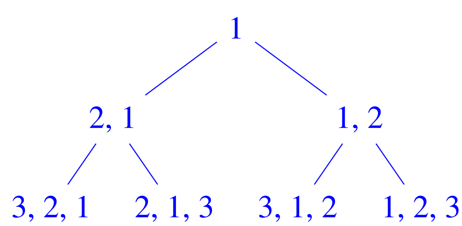
N people have come to watch a play and were given a row with exactly N consecutive seats. They have decided on the following seating arrangement. After the first person sits down, the next person has to sit next to the first. The third sits next to one of the first two and so on until all N are seated. In other words, no person can take a seat that separates him/her from at least one other person. How many different ways can this be accomplished? Note that the first person can choose any of the N seats. [*Hint*: Use induction.]

Solution:

Solution 1: The idea behind this solution is to count the number of orderings of students that can arise. Once an ordering of students is determined, there is only one possible way for the N students to sit among the N seats with this ordering. The first student can sit anywhere. For $k > 1$, the kth student can either choose to sit to the left or the right of all of the students already seated, for a total of 2 choices. Since there are 2 choices for each of the $N-1$ students (all students except the first student), the total number of choices is

$$2 \underbrace{\cdots}_{N-1 \text{ times}} 2 = 2^{N-1}.$$

We will discuss the multiplication of possibilities above in more detail when we discuss counting. For now, consider the following diagram for $N = 3$:



As you can see, there are $2^{3-1} = 4$ possible orderings.

Solution 2: We will prove the following statement by induction: for all $N \in \mathbb{N}$, there are 2^{N-1} ways to seat the students. The base case of $N = 1$ student is easy (there is only one way to seat the student). Suppose that the statement holds for k students; we will prove it holds for $k+1$ students. Note that the last student to arrive must sit either in the far left seat, or the far right seat. In either case, we must now seat k students in k seats, and by the inductive hypothesis, we know there are 2^{k-1} ways to achieve this. Therefore, the number of ways to seat $k+1$ students is

$$2^{k-1} + 2^{k-1} = 2 \cdot 2^{k-1} = 2^k,$$

so the statement holds for $k+1$. By the principle of induction, we are done.

3 Well-Ordered Grid

Consider an infinite sheet of graph paper such that each square contains a natural number. Suppose that the number in each square is equal to the average of the numbers in the four neighboring squares.

(a) By the Well-Ordering Principle, there must be some smallest number in the grid (call it n). Prove that for any square containing n, the four squares adjacent to it must also contain n.

(b) Prove that each square in the infinite grid contains the same number.

Solution:

(a) The basic idea is, the numbers around the minimum number n can't be larger, because that would raise the average above the n.
Formally: Suppose the four adjacent squares contain the numbers $w,x,y,$ and z. For the sake of contradiction, assume that one of the four numbers (say w) is not equal to n. By our choice of n as the smallest number, we have $n \leq w,x,y,z$ and in fact we have $n < w$. Then

$$\frac{w+x+y+z}{4} > \frac{n+x+y+z}{4} \geq \frac{n+n+n+n}{4} = n.$$

But since each square is equal to the average of the squares around it, the LHS is equal to n, a contradiction.

(b) Introduce this problem by drawing out a finite grid, with the minimum number n in the center. Follow the chain of implications: First the 4 immediately adjacent cells must be equal to n. Then, the cells around those must also be equal to n, etc. Then move onto formalizing this by induction:

Let S be a square which contains the number n. Let us introduce the following definition: we say that a square T is of distance d from S if we can find a chain of $d+1$ adjacent squares which begins at S and ends at T. Now let T be any square in the grid. We must show that T contains the number n. The proof will be by induction on d, the distance from T to S.
Base Case: The base case occurs when $d = 0$, namely when $T = S$. Then there is nothing to prove, since S contains the number n by definition.
Inductive hypothesis: Assume that each square of distance d from S contains the number n.
Inductive Step: Suppose T is of distance $d+1$ from S. Then one of the four squares adjacent to T, say U, is of distance d from S. By the inductive hypothesis, U contains the number n. Now by part (a), all squares neighboring U must also contain n, so in particular T must contain n.

4 Elephant Mosquito Paradox

Claim: The weight of an elephant equals the weight of a mosquito.

Proof: Let x be the weight of an elephant, and y that of a mosquito. Call the sum of the two weights $2v$, so that

$$x + y = 2v.$$

From this equation we can obtain two more.

$$x - 2v = -y, x = -y + 2v$$

Multiplying those together, we get

$$x^2 - 2vx = y^2 - 2vy.$$

Add v^2 to both sides.

$$x^2 - 2vx + v^2 = y^2 - 2vy + v^2$$
$$(x-v)^2 = (y-v)^2$$

Taking square roots, we get

$$x - v = y - v.$$

From this we conclude: $x = y$. That is, the elephant's weight (x) equals the mosquito's weight (y). Q.E.D. What is wrong here? You only need to find one wrong step, but identify all the wrong steps if you find more than one.

Solution:

When taking the square root of both sides, we have to add absolute value bars to both sides. Instead of $x - v = y - v$, we should then have

$$|x - v| = |y - v|.$$

Intuitively, v is the average, given that $v = (x + y)/2$. As a result, it is clear that both x and y are the same distance away from v. From there, we can see that the trivial statement $|x - v| = |y - v|$ is just misconstrued $x - v = y - v$.