CS 70 Discrete Mathematics and Probability Theory
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1 Well-Ordering Principle

In this question, we will go over how the well-ordering principle can be derived from (strong) induction. Remember the well-ordering principle states the following:

For every non-empty subset S of the set of natural numbers $\mathbb{N}$, there is a smallest element $x \in S$; i.e.

$$\exists x : \forall y \in S : x \leq y.$$

(a) What is the significance of S being non-empty? Does WOP hold without it? Assuming that S is not empty is equivalent to saying that there exists some number z in it.

(b) Induction is always stated in terms of a property that can only be based on a natural number. What should the induction be based on? The length of the set S? The number x? The number y? The number z?

(c) Now that the induction variable is clear, state the induction hypothesis. Be very precise. Do not leave out dangling symbols other than the induction variable. Ideally you should be able to write this in mathematical notation.

(d) Verify the base case. Note that your base case does not just consist of a single set S.

(e) Now prove that the induction works, by writing the inductive step.

(f) What should you change so that the proof works by simple induction (as opposed to strong induction)?

Solution:

(a) If S is empty, then WOP does not hold obviously. The significance is that with S not empty, you can always take a number out of it and start from there.

(b) The induction is based on z—an arbitrary natural number. By induction, we aim to prove our statement for all such z.

(c) **Hypothesis:** Assume that all sets S containing an element $0 \leq s \leq z$ contain a smallest element. In pure mathematical notation, this is: $\forall S \subseteq \mathbb{N}, (\exists s \in S,\ 0 \leq s \leq z) \implies \exists x : \forall y \in S : x \leq y$

(d) **Base Case:** For $z = 0$, the claim is true. Because we can take $x = 0$ as the smallest element and for all $y \in S$ we have $y \in \mathbb{N}$, and therefore $y \geq 0 = x$.

(e) **Inductive Step:** Let S be a set that contains $z+1$. If $z+1$ is the smallest element, we are done. Otherwise there exists $y \in S$, such that $y \leq z$. But now, by the inductive hypothesis, we know that S contains a smallest element.

(f) In order to make the proof work without strong induction, one can modify the induction hypothesis in the following way: **Hypothesis (in terms of** z**)** For all sets S that contain a number z' such that $z' \leq z$, the set S contains a smallest element.

This makes the proof work, simply because when the set contains a smaller element than z, we know that the smaller element is less than or equal to $z-1$, which allows us to appeal to the inductive hypothesis for $z-1$. Otherwise, if there are no smaller elements than z, z must be the smallest element.

Note: You may notice or be confused by the fact that the inductive hypotheses read almost exactly the same. This is because "weak" and "strong" induction are labels for how we frame the problem.

Recall that in strong induction the inductive hypothesis takes on the form:

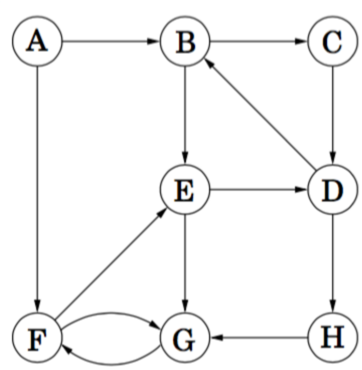
$$\forall 0 \leq i \leq k, P(i)$$

(thus we are assuming $P(0) \cdots P(k)$) while in weak induction the inductive hypothesis is simply $Q(k)$. So we can always define $Q(k) := \forall 0 \leq i \leq k, P(i)$ to convert strong induction to weak induction.

In this problem, in our "strong" inductive hypothesis we assumed $\forall 0 \leq s \leq z, P(s)$ for $P(s)$ defined to be "any set S containing an element s contains a smallest element", while in our "weak" inductive hypothesis we assumed $Q(z)$ for $Q(z)$ defined to be "any set S containing any element $0 \leq s \leq z$ contains a smallest element".

2 Graph Basics

In the first few parts, you will be answering questions on the following graph G.



(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

i. $\{(B,C),(C,D),(D,B)\}$

ii. $\{(F,G),(G,F)\}$

iii. $\{(A,B),(B,C),(C,D),(D,B)\}$

iv. $\{(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)\}$

(e) Which of the following are walks in G?

i. $\{(E,G)\}$

ii. $\{(E,G),(G,F)\}$

iii. $\{(F,G),(G,F)\}$

iv. $\{(A,B),(B,C),(C,D)\}$

v. $\{(E,G),(G,F),(F,G),(G,F)\}$

vi. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

(f) Which of the following are tours in G?

i. $\{(E,G)\}$

ii. $\{(E,G),(G,F)\}$

iii. $\{(F,G),(G,F)\}$

iv. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

Solution:

(a) A graph is specified as an ordered pair $G = (V,E)$, where V is the vertex set and E is the edge set.

$$V = \{A,B,C,D,E,F,G,H\},$$
$$E = \{(A,B),(A,F),(B,C),(B,E),(C,D),(D,B),(D,H),(E,D),(E,G),(F,E),(F,G),(G,F),(H,G)\}.$$

(b) G has the highest in-degree (3). A has the lowest in-degree (0).

$\{B,C,D,E,F,H\}$ all have the same in-degree and out-degree. H and C has in-degree (out-degree) equal to 1 and the other four have in-degree (out-degree) equal to 2.

(c) There are three paths:
$\{(B,C),(C,D),(D,H),(H,G),(G,F)\}$ (length = 5)
$\{(B,E),(E,D),(D,H),(H,G),(G,F)\}$ (length = 5)
$\{(B,E),(E,G),(G,F)\}$ (length = 3)

The last one listed above has the shortest path.

(d) A cycle should be a path that starts and ends at the same point, so iii is not a cycle. In addition, all the vertices $\{v_1,\ldots,v_n\}$ in the cycle $\{(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n)\}$ should be distinct, so iv is not a cycle. The correct answers are i and ii.

(e) All of them. A walk can end on the same vertex on which it begins or on a different vertex. A walk can travel over any edge and any vertex any number of times.

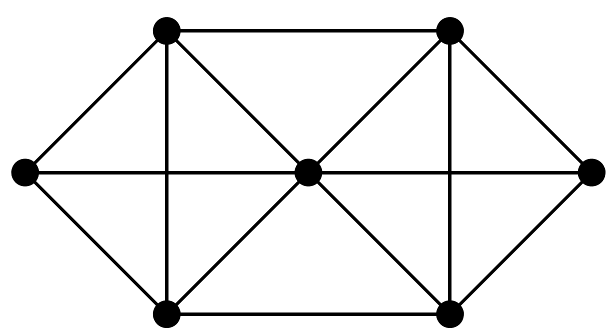
(f) iii. A tour is a walk which starts and ends at the same vertex.

(g) True. In a cycle, every vertex has degree at least 2. The most obvious sense in which one can see this is the fact that a cycle starts and ends at some vertex v, which would need to have a degree of 2 at least to account for both edges.

(h) True. Consider starting a walk at some vertex v_0, and at each step, walking along a previously untraversed edge, stopping when we first visit some vertex w for the second time. If such a process succeeds, then the part of our walk from the first time we visited w until the second time is a cycle, so it remains only to argue this process succeeds. Each time we take a step from some vertex v, since we are not stopping, we must have visited that vertex exactly once and not yet left. It follows that we have used at most one edge incident with v (either we started at v, or we took an edge into v). Since v has degree at least 2, there must be another edge leaving v for us to take.

(i) False. For example, a 3-cycle (triangle) is connected and every vertex has degree 2.

3 Eulerian Tour and Eulerian Walk



1. Is there an Eulerian tour in the graph above?

2. Is there an Eulerian walk in the graph above?

3. What is the condition that there is an Eulerian walk in an undirected graph?

Solution:

1. No. Two vertices have odd degree.

2. Yes. One of the two vertices with odd degree must be the first vertex, and the other one must be the last vertex.

3. An undirected graph has an Eulerian walk if and only if it is connected (except for isolated vertices) and at most two vertices have odd degree.

   Note: There is no graph with only one odd degree vertex.

4 Odd Degree Vertices

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$.

(ii) Induction on $m = |E|$ (number of edges)

(iii) Induction on $n = |V|$ (number of vertices)

(iv) Well-ordering principle

Solution:

Let $V_{\text{odd}}(G)$ denote the set of vertices in G that have odd degree. We prove that $|V_{\text{odd}}(G)|$ is even.

(i) Let d_v denote the degree of vertex v (so $d_v = |N_v|$, where N_v is the set of neighbors of v). Observe that

$$\sum_{v \in V} d_v = 2m$$

because every edge is counted exactly twice when we sum the degrees of all the vertices. Now partition V into the odd degree vertices $V_{\text{odd}}(G)$ and the even degree vertices $V_{\text{odd}}(G)^{\mathsf{c}}$, so we can write

$$\sum_{v \in V_{\text{odd}}(G)} d_v = 2m - \sum_{v \notin V_{\text{odd}}(G)} d_v.$$

Both terms in the right-hand side above are even ($2m$ is even, and each term d_v is even because we are summing over even degree vertices $v \notin V_{\text{odd}}(G)$). So for the left-hand side

$\sum_{v \in V_{\text{odd}}(G)} d_v$ to be even, we must have an even number of terms, since each term in the summation is odd. Therefore, there must be an even number of odd-degree vertices, namely, $|V_{\text{odd}}(G)|$ is even.

(ii) We use induction on $m \geq 0$.

Base case $m = 0$: If there are no edges in G, then all vertices have degree 0, so $V_{\text{odd}}(G) = \varnothing$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with m edges.

Inductive step: Let G be a graph with $m+1$ edges. Remove an arbitrary edge $\{u,v\}$ from G, so the resulting graph G' has m edges. By the inductive hypothesis, we know $|V_{\text{odd}}(G')|$ is even. Now add the edge $\{u,v\}$ to get back the original graph G. Note that u has one more edge in G than it does in G', so $u \in V_{\text{odd}}(G)$ if and only if $u \notin V_{\text{odd}}(G')$. Similarly, $v \in V_{\text{odd}}(G)$ if and only if $v \notin V_{\text{odd}}(G')$. The degrees of all other vertices are unchanged in going from G' to G. Therefore,

$$V_{\text{odd}}(G) = \begin{cases} V_{\text{odd}}(G') \cup \{u,v\} & \text{if } u,v \notin V_{\text{odd}}(G') \\ V_{\text{odd}}(G') \setminus \{u,v\} & \text{if } u,v \in V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{u\}) \cup \{v\} & \text{if } u \in V_{\text{odd}}(G'), v \notin V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{v\}) \cup \{u\} & \text{if } u \notin V_{\text{odd}}(G'), v \in V_{\text{odd}}(G') \end{cases}$$

so we see that $|V_{\text{odd}}(G)| - |V_{\text{odd}}(G')| \in \{-2, 0, 2\}$. Since $|V_{\text{odd}}(G')|$ is even, we conclude $|V_{\text{odd}}(G)|$ is also even.

(iii) We use induction on $n \geq 1$.

Base case $n = 1$: If G only has 1 vertex, then that vertex has degree 0, so $V_{\text{odd}}(G) = \varnothing$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with n vertices.

Inductive step: Let G be a graph with $n+1$ vertices. Remove a vertex v and all edges adjacent to it from G. The resulting graph G' has n vertices, so by the inductive hypothesis, $|V_{\text{odd}}(G')|$ is even. Now add the vertex v and all edges adjacent to it to get back the original graph G. Let $N_v \subseteq V$ denote the neighbors of v (i.e., all vertices adjacent to v). Among the neighbors N_v, the vertices in the intersection $A = N_v \cap V_{\text{odd}}(G')$ had odd degree in G', so they now have even degree in G. On the other hand, the vertices in $B = N_v \cap V_{\text{odd}}(G')^{\mathsf{c}}$ had even degree in G', and they now have odd degree in G. The vertex v itself has degree $|N_v|$, so $v \in V_{\text{odd}}(G)$ if and only if $|N_v|$ is odd. We now consider two cases:

(a) Suppose $|N_v|$ is even, so $v \notin V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B$$

so $|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B|$. Note that A and B are disjoint and their union equals N_v, so $|A| + |B| = |N_v|$. Therefore, we can write $|V_{\text{odd}}(G)|$ as

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| + |N_v| - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is even by assumption.

(b) Suppose $|N_v|$ is odd, so $v \in V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B \cup \{v\}$$

so, again using the relation $|A| + |B| = |N_v|$, we can write

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B| + 1 = |V_{\text{odd}}(G')| + (|N_v| + 1) - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is odd by assumption.

This completes the inductive step and the proof.

Note how this proof is more complicated than the proof in part (ii), even though they are both using induction. This tells you that choosing the right variable to induct on can simplify the proof.

(iv) Here we give a well-ordering proof using the number of edges m as the notion of "size" of G, so this is equivalent to the proof in part (ii) using induction on m. (You can also try to give a well-ordering proof using n as the size of G.)

Suppose the contrary that the claim is false for some graphs. This means the set M is not empty, where M is the set of $m \in \mathbb{N}$ for which there exists a graph G with m edges that is a counterexample to the claim. Thus, we have a nonempty subset M of $\mathbb{N}$, so by the well-ordering principle, M has a smallest element m'. Note that $m' > 0$, since the claim is true for all graphs with 0 edges.

Let G be a graph with m' edges for which the claim is false, i.e., $|V_{\text{odd}}(G)|$ is odd (here we know such a G must exist from the definition of $m' \in M$). Remove one edge from G to obtain a smaller graph G' with $m' - 1$ edges (here we need $m' \geq 1$, which we have seen above). By our choice of m' as the smallest element of M, we know that $m' - 1 \notin M$, so the claim holds for G', namely, $|V_{\text{odd}}(G')|$ is even. Now add the removed edge to get back G. By the same argument as in the inductive step in part (ii), this implies that $|V_{\text{odd}}(G)|$ is also even, a contradiction.