1 Well-Ordering Principle

In this question, we will go over how the well-ordering principle can be derived from (strong) induction. Remember the well-ordering principle states the following:

For every non-empty subset S of the set of natural numbers $\mathbb{N}$, there is a smallest element $x \in S$; i.e.

$$\exists x : \forall y \in S : x \leq y.$$

(a) What is the significance of S being non-empty? Does WOP hold without it? Assuming that S is not empty is equivalent to saying that there exists some number z in it.

(b) Induction is always stated in terms of a property that can only be based on a natural number. What should the induction be based on? The length of the set S? The number x? The number y? The number z?

(c) Now that the induction variable is clear, state the induction hypothesis. Be very precise. Do not leave out dangling symbols other than the induction variable. Ideally you should be able to write this in mathematical notation.

(d) Verify the base case. Note that your base case does not just consist of a single set S.

(e) Now prove that the induction works, by writing the inductive step.

(f) What should you change so that the proof works by simple induction (as opposed to strong induction)?

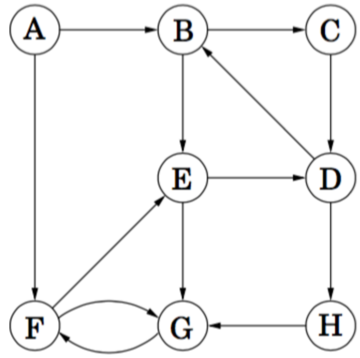
2 Graph Basics

In the first few parts, you will be answering questions on the following graph G.

(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?



(d) Which of the following are cycles in G?

i. $\{(B,C),(C,D),(D,B)\}$
ii. $\{(F,G),(G,F)\}$
iii. $\{(A,B),(B,C),(C,D),(D,B)\}$
iv. $\{(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)\}$

(e) Which of the following are walks in G?

i. $\{(E,G)\}$
ii. $\{(E,G),(G,F)\}$
iii. $\{(F,G),(G,F)\}$
iv. $\{(A,B),(B,C),(C,D)\}$
v. $\{(E,G),(G,F),(F,G),(G,F)\}$
vi. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

(f) Which of the following are tours in G?

i. $\{(E,G)\}$
ii. $\{(E,G),(G,F)\}$
iii. $\{(F,G),(G,F)\}$
iv. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

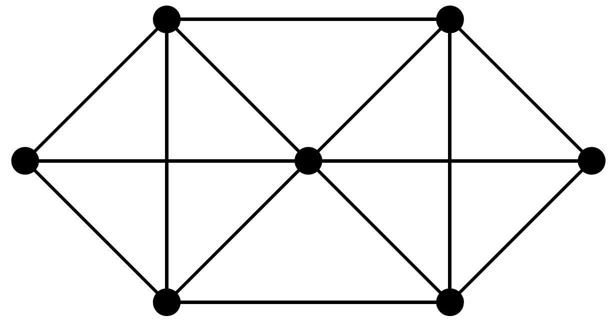
In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

3 Eulerian Tour and Eulerian Walk



1. Is there an Eulerian tour in the graph above?
2. Is there an Eulerian walk in the graph above?
3. What is the condition that there is an Eulerian walk in an undirected graph?

4 Odd Degree Vertices

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that* $\sum_{v \in V} \deg v = 2|E|$.

(ii) Induction on $m = |E|$ (number of edges)

(iii) Induction on $n = |V|$ (number of vertices)

(iv) Well-ordering principle