DIS 2A

1 Stable Marriage

Consider the set of men $M = \{1,2,3\}$ and the set of women $W = \{A,B,C\}$ with the following preferences.

Men	Women		
1	A	B	C
2	B	A	C
3	A	B	C

Women	Men		
A	2	1	3
B	1	2	3
C	1	2	3

Run the male propose-and-reject algorithm on this example. How many days does it take and what is the resulting pairing? (Show your work.)

Solution:

The algorithm takes 3 days to produce a matching. The resulting pairing is as follows. The circles indicate the man that a woman picked on a given day (and rejected the rest).

$$\{(A,1),(B,2),(C,3)\}.$$

Woman	Day 1	Day 2	Day 3
A	1,3	1	1
B	2	2,3	2
C			3

2 Stable Marriage

The following questions refer to stable marriage instances with n men and n women, answer True/-False or provide an expression as requested.

(a) For $n = 2$, or any 2-man, 2-woman stable marriage instance, man A has the same optimal and pessimal woman. (True or False.)

(b) In any stable marriage instance, in the pairing the TMA produces there is some man who gets his favorite woman (the first woman on his preference list). (True or False.)

(c) In any stable marriage instance with n men and women, if every man has a different favorite woman, a different second favorite, a different third favorite, and so on, and every woman has the same preference list, how many days does it take for TMA to finish? (Form of Answer: An expression that may contain n.)

(d) Consider a stable marriage instance with n men and n women, and where all men have the same preference list, and all women have different favorite men, and different second-favorite men, and so on. How many days does the TMA take to finish? (Form of Answer: An expression that may contain n.)

(e) It is possible for a stable pairing to have a man A and a woman 1 be paired if A is 1's least preferred choice and 1 is A's least preferred choice. (True or False.)

(f) It is possible for a stable pairing to have two couples where each person is paired with their least favorite choice. (True or False.)

(g) If there is a pairing, P, that consists of only pairs from male and female optimal pairings, then it must be stable. In other words, if every pair in P is a pair either in the male-optimal or the female-optimal pairing then P is stable. (True or False.)

Solution:

(a) **False.** This says there is only one stable pairing. But if the preference list for man A is $(1,2)$ and for man B is $(2,1)$ and preference list for woman 1 is (B,A) and woman 2 is (A,B) produce different male and female optimal pairings.

(b) **False.** Let man A have preference list $(1,3,2)$, B have $(1,2,3)$, and C have $(2,1,3)$. We develop a "cyclic" chain of preferences, causing A to displace B to displace C who then displaces A.

   (a) If woman 1 prefers A over B, she puts A on a string and rejects B.

   (b) B does not get his favorite and proposes to 2, who prefers B over C and thus rejects C.

   (c) C does not get his favorite and proposes to 1, who prefers C over A and thus rejects A.

   Thus, A also does not get his favorite, and no man gets his favorite.

(c) **1**.
On the first day every woman gets a proposal since each man has a different woman in their first position. The algorithm terminates.

(d) **n**.
Every man proposes to their common favorite. One man is kept on the string. The rest propose to the second. And so one. After each day, a new woman gets a man on a string. After n days, we finish. Note: that the men's preference lists (assuming they're the same for everyone) were irrelevant.

(e) **True.**
A and 1 are respectively all the women's and men's least favorite.

(f) **False.**

Consider the two-couple case. The man from the first and the woman from the other prefer each other, thus they form a rogue couple.

(g) **False**.

Consider a woman who is matched to her pessimal partner and a man who is matched to his pessimal partner. They may well like each other.

An example is as follows.

Men's preference list
A: $1 > \ldots > 2$
B: $2 > \ldots > 1$
C: $3 > \ldots > 4$
D: $4 > \ldots > 3$

Women's preference list
1: $B > \ldots > A$
2: $A > \ldots > B$
3: $D > \ldots > C$
4: $C > \ldots > D$

Men's first choices = women's last choices and vice versa.

men-optimal: $(A,1)$, $(B,2)$, $(C,3)$, $(D,4)$
women-optimal: $(B,1)$, $(A,2)$, $(D,3)$, $(C,4)$
our pairing: $(A,1)$, $(B,2)$, $(D,3)$, $(C,4)$ and $(C,1)$ is a rogue couple.

3 Universal Preference

Suppose that preferences in a stable marriage instance are universal: all n men share the preferences $W_1 > W_2 > \cdots > W_n$ and all women share the preferences $M_1 > M_2 > \cdots > M_n$.

(a) What result do we get from running the algorithm with men proposing? Can you prove it?

(b) What result do we get from running the algorithm with women proposing?

(c) What does this tell us about the number of stable matchings?

Solution:

(a) The pairing results in (W_i, M_i) for each $i \in \{1, 2, ..., n\}$.

This result can be proved by induction:

Base case: when $n = 1$, the only pairing is (W_1, M_1), and the base case is true.
Now assume this is true for some $n \in \mathbb{N}$.
On the first day with $n+1$ men and $n+1$ women, all $n+1$ men will propose to W_1. W_1 prefers M_1 the most, and the rest of the men will be rejected. This leaves a set of n unpaired men and n unpaired women who all have the same preferences (after the pairing of (W_1, M_1)). By the process of induction, this means that every i^{th} preferred woman will be paired with the i^{th} preferred man.

(b) The pairings will again result in (M_i, W_i) for each $i \in \{1, 2, ..., n\}$. This can be proved by induction in the same as above, but replacing "man" with "woman" and vice-versa.

(c) We know that male-proposing produces a female-pessimal stable pairing. We also know that female-proposing produces a female-optimal stable pairing. We found that female-optimal and female-pessimal pairings are the same. This means that there is only one stable pairing, since both the best and worst pairings (for females) are the same pairings.