1 Stable Marriage

Consider the set of men $M = \{1, 2, 3\}$ and the set of women $W = \{A, B, C\}$ with the following preferences.

Men	Women		
1	A	B	C
2	B	A	C
3	A	B	C

Women	Men		
A	2	1	3
B	1	2	3
C	1	2	3

Run the male propose-and-reject algorithm on this example. How many days does it take and what is the resulting pairing? (Show your work)

Solution:

The algorithm takes 3 days to produce a matching. The resulting pairing is

$$\{(A,1),(B,2),(C,3)\}.$$

Woman	Day 1	Day 2	Day 3
A	1,3	1	1
B	2	2,3	2
C			3

2 Quantitative Stable Marriage Algorithm

Once you have practiced the basic algorithm, let's quantify stable marriage problem a little bit. Here we define the following notation: on day j, let $P_j(M)$ be the rank of the woman that man M proposes to (where the first woman on his list has rank 1 and the last has rank n). Also, let $R_j(W)$ be the total number of men that woman W has rejected up through day $j-1$ (i.e. not including the proposals on day j). Please answer the following questions using the notation above.

(a) Prove or disprove the following claim: $\sum_M P_j(M) - \sum_W R_j(W)$ is independent of j. If it is true, please also give the value of $\sum_M P_j(M) - \sum_W R_j(W)$. The notation, $\sum_M$ and $\sum_W$, simply means that we are summing over all men and all women.

(b) Prove or disprove the following claim: one of the **men or women** must be matched to someone who is ranked in the top half of their preference list. You may assume that n is even.

Solution:

(a) On day $j = 1$, each man proposes to the first woman on his list so $\sum_M P_1(M) = n$, and no woman rejected any man through day 0, and therefore $\sum_M P_1(M) - \sum_W R_1(W) = n$. In general, each time a woman rejects a man on day $j-1$, it increases $\sum_W R_j(W)$ by exactly 1. It also increases $\sum_M P_j(M)$ by exactly 1, since the rejected man proposes to the next woman on his list on day j. Therefore $\sum_M P_j(M) - \sum_W R_j(W)$ stays constant and is independent of j. $\square$

More formally, we can prove this by induction on j, with $j = 1$ as base case.
Induction Hypothesis: Assume $\sum_M P_j(M) - \sum_W R_j(W) = n$.
Induction Step: The quantity $\sum_W R_{j+1}(W) - \sum_W R_j(W)$ is exactly the number of men rejected by women on day j. But each of the rejected men propose to the next woman on their list on day $j+1$, and so this quantity is also equal to $\sum_M P_{j+1}(M) - \sum_M P_j(M)$. Equating the two, we get

$$\sum_W R_{j+1}(W) - \sum_W R_j(W) = \sum_M P_{j+1}(M) - \sum_M P_j(M).$$

Therefore,

$$\sum_M P_{j+1}(M) - \sum_W R_{j+1}(W) = \sum_M P_j(M) - \sum_W R_j(W)$$

and the right hand side is equal to n by the induction hypothesis. $\square$

(b) Assume that no man is matched with a woman in the top half of his preference list. Each of them must have been rejected at least $n/2$ times, for a total of at least $n^2/2$ rejections. This implies that at least one woman must have rejected at least $n/2$ men (because if not, then the total number of rejections must be less than $(n/2) \cdot n$, contradiction). But now, by the improvement lemma, this woman must be matched with a man she likes more than the $n/2$ men she rejected, meaning that the man she is matched with is in the top half of her preference list. $\square$

Alternative Proof:

Assume towards contradiction that every man and every woman is matched to someone who is ranked in the bottom half of their preference list.

Observe that a man M is matched to someone in the top half of his preference list if and only if $P_m(M) \leq n/2$, where m is the last day of the algorithm. Therefore, if M is matched to someone in the bottom half of his preference list, then $P_m(M) > n/2$, i.e., $P_m(M) \geq n/2 + 1$. Summing over the men gives us $\sum_M P_m(M) \geq n^2/2 + n$. By part (a), it follows that $\sum_W R_m(W) = \sum_M P_m(M) - n \geq n^2/2$.

Observe also that if $R_m(W) \geq n/2$, then by the improvement lemma, W must be matched to someone in the top half of her preference list. Therefore, from our assumption that W is matched to someone in the bottom half of her preference list, we get $R_m(W) < n/2$. Summing over the women gives us $\sum_W R_m(W) < n^2/2$. But this contradicts our earlier result above! $\square$

3 Be a Judge

For each of the following statements about the traditional stable marriage algorithm with men proposing, indicate whether the statement is True or False and justify your answer with a short 2-3 line explanation:

(a) There is a set of preferences for n men and n women, such that in a stable marriage algorithm execution every man ends up with his least preferred woman.

(b) In a stable marriage instance, if man M and woman W each put each other at the top of their respective preference lists, then M must be paired with W in every stable pairing.

(c) In a stable marriage instance with at least two men and two women, if man M and woman W each put each other at the bottom of their respective preference lists, then M cannot be paired with W in any stable pairing.

(d) For every $n > 1$, there is a stable marriage instance with n men and n women which has an unstable pairing in which every unmatched man-woman pair is a rogue couple.

Solution:

(a) **False**: If this were to occur it would mean that at the end of the algorithm, every man would have proposed to every woman on his list and has been rejected $n-1$ times. This would also require every woman to reject $n-1$ suitors. We know this is impossible though, as we learned above that at least one woman receives a single proposal. There must be at least one woman who is not proposed to until the very last day.

(b) **True**: We give a simple proof by contradiction. Assume that M and W can put each other at the top of their respective preference lists, but M and W are not paired with each other in some stable pairing. Then we have a stable pairing which includes the pairings (M, W'), (M', W), for some man M' and woman W'. However, M prefers W over his partner in this pairing, since W is at the top of his preference list. Similarly W prefers M over her partner. Thus (M,W) form a rogue couple, so the pairing is not stable. We have arrived at a contradiction.

Therefore if man M and woman W put each other at the top of their respective preference lists, then M must be paired with W in a stable pairing.

(c) **False**: The key here is to realize that this is possible if man M and woman W are at the bottom of everybody else's preference list as well. Consider the following example with the men m and M and the women w and W. Suppose that their preference lists are as follows:

$$\begin{aligned} m &: w, W \\ M &: w, W \\ w &: m, M \\ W &: m, M \end{aligned}$$

It is clear that M and W are at the bottom of each other's preference lists; however, it is also true that (m,w) and (M,W) is a stable pairing (indeed, it is the only stable pairing). So, we have a contradiction to the statement: here is a stable marriage instance with at least two men and two women, and man M and woman W put each other at the bottom of their respective preference lists, but yet M and W are paired together in a stable pairing.

(d) **True**: Suppose $n > 1$ and we have men $M_1, \dots, M_n$ and women $W_1, \dots, W_n$. Further, assume that for $1 \le i \le n$, preference lists are as follows for every man M_i and woman W_i:

$$\begin{array}{rl} & \textit{highest} \Longrightarrow \textit{lowest} \\ M_i: & W_i \;\; W_{i+1} \;\; W_{i+2} \;\; \dots \;\; W_{i-1} \end{array}$$

$$\begin{array}{rl} & \textit{highest} \Longrightarrow \textit{lowest} \\ W_i: & M_i \;\; M_{i-1} \;\; M_{i-2} \;\; \dots \;\; M_{i+1} \end{array}$$

Note that the indices are taken modulo n, so if i refers to $n+1$ in the preference lists above, it is really referring to 1. The idea in this construction is that there is a fixed ordering of men into a cycle, and a fixed ordering of women into another cycle. Every man's preference list complies to the ordering of women into the cycle, with the only difference between different men's preferences being where in the ordering the preference list begins. The analogous situation holds for women's preference lists.

Now consider the unstable pairing in which each man M_i, $1 \le i \le n$ is paired as (M_i, W_{i-1}). (M_1 is paired to W_n.) We claim every unmatched man-woman pair is a rogue couple.

In this pairing, every man M_i is paired with woman W_{i-1} at the bottom of his preference list, and every woman W_i is paired with man M_{i+1} at the bottom of her preference list. Thus every man prefers any woman he has not been matched to over his partner, and likewise for women. So any unmatched pair (M,W) is a rogue couple.

4 Universal Preference

Suppose that preferences in a stable marriage instance are universal: all n men share the preferences $W_1 > W_2 > \cdots > W_n$ and all women share the preferences $M_1 > M_2 > \cdots > M_n$.

(a) What result do we get from running the algorithm with men proposing? Can you prove it?

(b) What result do we get from running the algorithm with women proposing?

(c) What does this tell us about the number of stable matchings?

Solution:

(a) The pairing results in (W_i, M_i) for each $i \in \{1, 2, \dots, n\}$.
We can running the propose and reject algorithm to let students know what actually happens while running the algorithm if they all have the same preference. This result can be proved by induction:

Base case: when $n = 1$, the only pairing is (W_1, M_1), and the base case is true.
Now assume this is true for some $n \in \mathbb{N}$.
On the first day with $n+1$ men and $n+1$ women, all $n+1$ men will propose to W_1. W_1 prefers M_1 the most, and the rest of the men will be rejected. This leaves a set of n unpaired men and n unpaired women who all have the same preferences (after the pairing of (W_1, M_1)). By the process of induction, this means that every i^{th} preferred woman will be paired with the i^{th} preferred man.

(b) The pairings will again result in (M_i, W_i) for each $i \in \{1, 2, \ldots, n\}$. This can be proved by induction in the same as above, but replacing "man" with "woman" and vice-versa.

(c) We know that male-proposing produces a female-pessimal stable pairing. We also know that female-proposing produces a female-optimal stable pairing. We found that female-optimal and female-pessimal pairings are the same. This means that there is only one stable pairing, since both the best and worst pairings (for females) are the same pairings.