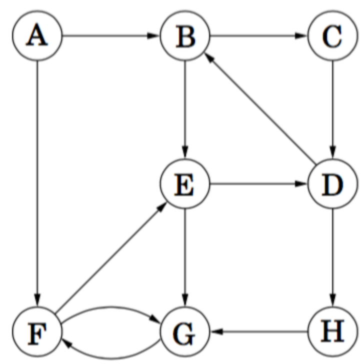
1 Graph Basics

In the first few parts, you will be answering questions on the following graph G.



(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

i. $\{(B,C),(C,D),(D,B)\}$
ii. $\{(F,G),(G,F)\}$
iii. $\{(A,B),(B,C),(C,D),(D,B)\}$
iv. $\{(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)\}$

(e) Which of the following are walks in G?

i. $\{(E,G)\}$
ii. $\{(E,G),(G,F)\}$
iii. $\{(F,G),(G,F)\}$
iv. $\{(A,B),(B,C),(C,D)\}$
v. $\{(E,G),(G,F),(F,G),(G,F)\}$
vi. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

(f) Which of the following are tours in G?

i. $\{(E,G)\}$

ii. $\{(E,G),(G,F)\}$

iii. $\{(F,G),(G,F)\}$

iv. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

Solution:

(a) A graph is specified as an ordered pair $G = (V,E)$, where V is the vertex set and E is the edge set.

$V = \{A,B,C,D,E,F,G,H\}$,
$E = \{(A,B),(A,F),(B,C),(B,E),(C,D),(D,B),(D,H),(E,D),(E,G),(F,E),(F,G),(G,F),$
$(H,G)\}$.

(b) G has the highest in-degree (3). A has the lowest in-degree (0).

$\{B,C,D,E,F,H\}$ all have the same in-degree and out-degree. H and C has in-degree (out-degree) equal to 1 and the other four have in-degree (out-degree) equal to 2.

(c) There are three paths:
$\{(B,C),(C,D),(D,H),(H,G),(G,F)\}$ (length = 5)
$\{(B,E),(E,D),(D,H),(H,G),(G,F)\}$ (length = 5)
$\{(B,E),(E,G),(G,F)\}$ (length = 3)

The last one listed above has the shortest path.

(d) A cycle should be a path that starts and ends at the same point, so iii is not a cycle. In addition, all the vertices $\{v_1,\ldots,v_n\}$ in the cycle $\{(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n)\}$ should be distinct, so iv is not a cycle. The correct answers are i and ii.

(e) All of them. A walk can end on the same vertex on which it begins or on a different vertex. A walk can travel over any edge and any vertex any number of times.

(f) iii. A tour is a walk which starts and ends at the same vertex.

(g) True. In a cycle, every vertex has degree at least 2. The most obvious sense in which one can see this is the fact that a cycle starts and ends at some vertex v, which would need to have a degree of 2 at least to account for both edges.

(h) True. Consider starting a walk at some vertex v_0, and at each step, walking along a previously untraversed edge, stopping when we first visit some vertex w for the second time. If such a process succeeds, then the part of our walk from the first time we visited w until the second time is a cycle, so it remains only to argue this process succeeds. Each time we take a step from some vertex v, since we are not stopping, we must have visited that vertex exactly once and not yet left. It follows that we have used at most one edge incident with v (either we started at v, or we took an edge into v). Since v has degree at least 2, there must be another edge leaving v for us to take.

(i) False. For example, a 3-cycle (triangle) is connected and every vertex has degree 2.

2 Odd Degree Vertices

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

(i) Direct proof (e.g., counting the number of edges in G)

(ii) Induction on $m = |E|$ (number of edges)

(iii) Induction on $n = |V|$ (number of vertices)

(iv) Well-ordering principle

Solution:

Let $V_{\text{odd}}(G)$ denote the set of vertices in G that have odd degree. We prove that $|V_{\text{odd}}(G)|$ is even.

(i) Let d_v denote the degree of vertex v (so $d_v = |N_v|$, where N_v is the set of neighbors of v). Observe that

$$\sum_{v \in V} d_v = 2m$$

because every edge is counted exactly twice when we sum the degrees of all the vertices. Now partition V into the odd degree vertices $V_{\text{odd}}(G)$ and the even degree vertices $V_{\text{odd}}(G)^{\mathsf{c}}$, so we can write

$$\sum_{v \in V_{\text{odd}}(G)} d_v = 2m - \sum_{v \notin V_{\text{odd}}(G)} d_v.$$

Both terms in the right-hand side above are even ($2m$ is even, and each term d_v is even because we are summing over even degree vertices $v \notin V_{\text{odd}}(G)$). So for the left-hand side $\sum_{v \in V_{\text{odd}}(G)} d_v$ to be even, we must have an even number of terms, since each term in the summation is odd. Therefore, there must be an even number of odd-degree vertices, namely, $|V_{\text{odd}}(G)|$ is even.

(ii) We use induction on $m \geq 0$.

Base case $m = 0$: If there are no edges in G, then all vertices have degree 0, so $V_{\text{odd}}(G) = \varnothing$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with m edges.

Inductive step: Let G be a graph with $m+1$ edges. Remove an arbitrary edge $\{u,v\}$ from G, so the resulting graph G' has m edges. By the inductive hypothesis, we know $|V_{\text{odd}}(G')|$ is even. Now add the edge $\{u,v\}$ to get back the original graph G. Note that u has one more edge in G than it does in G', so $u \in V_{\text{odd}}(G)$ if and only if $u \notin V_{\text{odd}}(G')$. Similarly, $v \in V_{\text{odd}}(G)$ if and only if $v \notin V_{\text{odd}}(G')$. The degrees of all other vertices are unchanged in going from G' to G. Therefore,

$$V_{\text{odd}}(G) = \begin{cases} V_{\text{odd}}(G') \cup \{u,v\} & \text{if } u,v \notin V_{\text{odd}}(G') \\ V_{\text{odd}}(G') \setminus \{u,v\} & \text{if } u,v \in V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{u\}) \cup \{v\} & \text{if } u \in V_{\text{odd}}(G'), v \notin V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{v\}) \cup \{u\} & \text{if } u \notin V_{\text{odd}}(G'), v \in V_{\text{odd}}(G') \end{cases}$$

so we see that $|V_{\text{odd}}(G)| - |V_{\text{odd}}(G')| \in \{-2,0,2\}$. Since $|V_{\text{odd}}(G')|$ is even, we conclude $|V_{\text{odd}}(G)|$ is also even.

(iii) We use induction on $n \geq 1$.

Base case $n = 1$: If G only has 1 vertex, then that vertex has degree 0, so $V_{\text{odd}}(G) = \varnothing$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with n vertices.

Inductive step: Let G be a graph with $n+1$ vertices. Remove a vertex v and all edges adjacent to it from G. The resulting graph G' has n vertices, so by the inductive hypothesis, $|V_{\text{odd}}(G')|$ is even. Now add the vertex v and all edges adjacent to it to get back the original graph G. Let $N_v \subseteq V$ denote the neighbors of v (i.e., all vertices adjacent to v). Among the neighbors N_v, the vertices in the intersection $A = N_v \cap V_{\text{odd}}(G')$ had odd degree in G', so they now have even degree in G. On the other hand, the vertices in $B = N_v \cap V_{\text{odd}}(G')^{\mathsf{c}}$ had even degree in G', and they now have odd degree in G. The vertex v itself has degree $|N_v|$, so $v \in V_{\text{odd}}(G)$ if and only if $|N_v|$ is odd. We now consider two cases:

(a) Suppose $|N_v|$ is even, so $v \notin V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B$$

so $|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B|$. Note that A and B are disjoint and their union equals N_v, so $|A| + |B| = |N_v|$. Therefore, we can write $|V_{\text{odd}}(G)|$ as

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| + |N_v| - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is even by assumption.

(b) Suppose $|N_v|$ is odd, so $v \in V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B \cup \{v\}$$

so, again using the relation $|A|+|B|=|N_v|$, we can write

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B| + 1 = |V_{\text{odd}}(G')| + (|N_v|+1) - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is odd by assumption.

This completes the inductive step and the proof.

Note how this proof is more complicated than the proof in part (ii), even though they are both using induction. This tells you that choosing the right variable to induct on can simplify the proof.

(iv) Here we give a well-ordering proof using the number of edges m as the notion of "size" of G, so this is equivalent to the proof in part (ii) using induction on m. (You can also try to give a well-ordering proof using n as the size of G.)

Suppose the contrary that the claim is false for some graphs. This means the set M is not empty, where M is the set of $m \in \mathbb{N}$ for which there exists a graph G with m edges that is a counterexample to the claim. Thus, we have a nonempty subset M of $\mathbb{N}$, so by the well-ordering principle, M has a smallest element m'. Note that $m' > 0$, since the claim is true for all graphs with 0 edges.

Let G be a graph with m' edges for which the claim is false, i.e., $|V_{\text{odd}}(G)|$ is odd (here we know such a G must exist from the definition of $m' \in M$). Remove one edge from G to obtain a smaller graph G' with $m'-1$ edges (here we need $m' \geq 1$, which we have seen above). By our choice of m' as the smallest element of M, we know that $m'-1 \notin M$, so the claim holds for G', namely, $|V_{\text{odd}}(G')|$ is even. Now add the removed edge to get back G. By the same argument as in the inductive step in part (ii), this implies that $|V_{\text{odd}}(G)|$ is also even, a contradiction.

3 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices, such that no 2 vertices in the same set have an edge between them. Consider an undirected bipartite graph with two disjoint sets L,R. Prove that a graph is bipartite if and only if it has no tours of odd length.

Solution:

Begin by proving the forward direction: an undirected bipartite graph has no tours of odd length.

Let us start traveling the tour from a node n_0 in L. Since each edge in the graph connects a vertex in L to one in R, the 1st edge in the tour connects our start node n_0 to a node n_1 in R. The 2nd edge in the tour must connect n_1 to a node n_2 in L. Continuing on, the $(2k+1)$-th edge connects node n_{2k} in L to node n_{2k+1} in R, and the $2k$-th edge connects node n_{2k-1} in R to node n_{2k} in L. Since only even numbered edges connect to vertices in L, and we started our tour in L, the tour must end with an even number of edges.

Prove the reverse direction: A undirected graph with no tours of odd length is bipartite.

Take some vertex v. Add all vertices where the shortest path to v is odd, to R. Add all vertices where the shortest path to v is even, to L. If any of the vertices in $u_1, u_2 \in R$ are connected, then we have a tour of odd length formed by appending: the shortest path between v and u_1 (odd), the edge (u_1, u_2) (odd), and the shortest path between u_2 and v (odd). This means no two vertices in R are connected. Similarly, if any two vertices $v_1, v_2 \in L$ are connected, we get a tour of odd length by appending: the shortest path between v and v_1 (even), the edge (v_1, v_2) (odd), and the shortest path between v_2 and v (even). This means no two vertices in L are connected either. If there are other connected componenets, we can proceed by choosing a new vertex in each component and repeating this process. Then we will have disjoint L, R which include all vertices.