1 Leaves in a Tree

A *leaf* in a tree is a vertex with degree 1.

(a) Consider a tree with $n \geq 3$ vertices. What is the largest possible number of leaves the tree could have? Prove that this maximum m is possible to achieve, and further that there cannot exist a tree with more than m leaves.

(b) Prove that every tree on $n \geq 2$ vertices must have at least two leaves.

Solution:

(a) We claim the maximum number of leaves is $n-1$. This is achieved when there is one vertex that is connected to all other vertices (this is called the *star graph*).

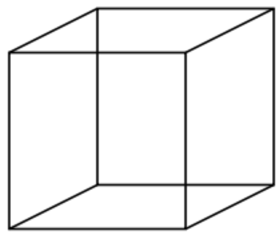
We now show that a tree on $n \geq 3$ vertices cannot have n leaves. Suppose the contrary that there is a tree on $n \geq 3$ vertices such that all its n vertices are leaves. Pick an arbitrary vertex x, and let y be its unique neighbor. Since x and y both have degree 1, the vertices x,y form a connected component separate from the rest of the tree, contradicting the fact that a tree is connected.

(b) We give a direct proof. Consider the longest path $\{v_0,v_1\},\{v_1,v_2\},\ldots,\{v_{k-1},v_k\}$ between two vertices $x = v_0$ and $y = v_k$ in the tree (here the length of a path is how many edges it uses, and if there are multiple longest paths then we just pick one of them). We claim that x and y must be leaves. Suppose the contrary that x is not a leaf, so it has degree at least two. This means x is adjacent to another vertex z different from v_1. Observe that z cannot appear in the path from x to y that we are considering, for otherwise there would be a cycle in the tree. Therefore, we can add the edge $\{z,x\}$ to our path to obtain a longer path in the tree, contradicting our earlier choice of the longest path. Thus, we conclude that x is a leaf. By the same argument, we conclude y is also a leaf.

The case when a tree has only two leaves is called the *path graph*, which is the graph on $V = \{1,2,\ldots,n\}$ with edges $E = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}$.

2 Cube Dual

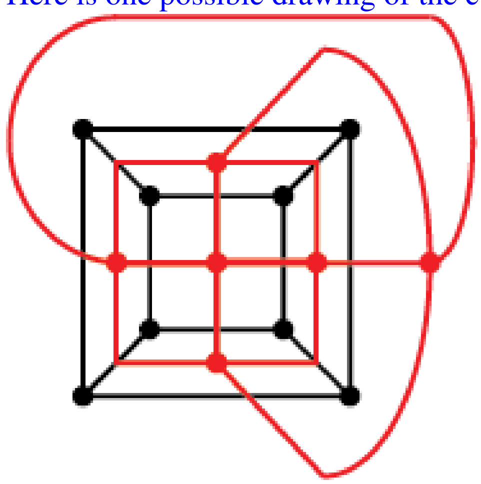
(a) We define a graph G by letting the vertices be the corners of a cube and having edges connecting adjacent corners. Draw a planar representation of G and the corresponding dual planar graph. (Below is a picture of a cube, provided for reference)



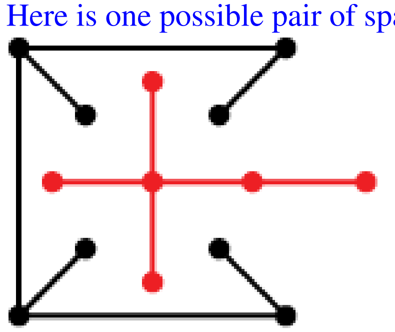
(b) Find a spanning tree of your planar drawing and identify the corresponding spanning tree of the dual planar graph.

Solution:

(a) Here is one possible drawing of the cube (in black) with its dual (in red):



(b) Here is one possible pair of spanning trees:



3 Planarity

Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Solution:

Assume G is planar. Take 5 vertices, they cannot form K_5, so some pair v_1, v_2 have no edge between them. The remaining five vertices of G cannot form K_5 either, so there is a second pair v_3, v_4 that have no edge between them. Now consider v_1, v_2 and any other three vertices v_5, v_6, v_7. Since $v_1 v_2$ is not an edge, by property T it must be that $v_1 v$ and $v_2 v$ where $v \in \{v_5, v_6, v_7\}$ are edges. Similarly for $v_3, v_4, v_3 v$ and $v_4 v$ where $v \in \{v_5, v_6, v_7\}$ are edges. So now any three vertices in $\{v_1, v_2, v_3, v_4\}$ on one side and $\{v_5, v_6, v_7\}$ on the other form an instance of $K_{3,3}$. Contradiction.

The above shows that any graph with 7 vertices and property T is non-planar. Any graph with > 7 vertices and property T will also be non-planar because it will contain a subgraph with 7 vertices and property T.

4 Planarity and Graph Complements

Let $G = (V, E)$ be a graph. We define the complement of G as $\overline{G} = (V, \overline{E})$ where $\overline{E} = (V \times V) - E$; that is, $\overline{G}$ has the same set of vertices as G, but an edge e exists is $\overline{G}$ if and only if it does not exist in G.

(a) Suppose G has v vertices and e edges. How many vertices and edges does $\overline{G}$ have?

(b) Prove that for any graph with at least 13 vertices, G being planar implies that $\overline{G}$ is non-planar.

(c) Is the converse of the previous part true? That is, if $\overline{G}$ is non-planar, does that imply that G is planar? Prove this or give a counterexample.

Solution:

(a) If G has v vertices, there are a total of $\frac{v(v-1)}{2}$ edges that could possibly exist in the graph. Since e of them do appear in G, we know that the remaining $\frac{v(v-1)}{2} - e$ of them must appear in $\overline{G}$.

(b) Since G is planar, we know that $e \leq 3v - 6$. Plugging this in to the answer from the previous part, we have that $\overline{G}$ has at least $\frac{v(v-1)}{2} - (3v - 6)$ edges. Since v is at least 13, we have that $\frac{v(v-1)}{2} \geq \frac{v \cdot 12}{2} = 6v$, so $\overline{G}$ has at least $6v - 3v + 6 = 3v + 6$ edges. Since this is strictly more than the $3v - 6$ edges allowed in a planar graph, we have that $\overline{G}$ must not be planar.

(c) The converse is not necessarily true. As a counterexample, suppose that G has exactly thirteen vertices, of which five are all connected to each other and the remaining eight have no edges incident to them. This means that G is non-planar, since it contains a copy of K_5. However, $\overline{G}$ also contains a copy of K_5 (take any 5 of the 8 vertices that were isolated in G), so $\overline{G}$ is also non-planar. Thus, it is possible for both G and $\overline{G}$ to be non-planar.