1 Trees

Recall that a *tree* is a connected acyclic graph (graph without cycles). In the note, we presented a few other definitions of a tree, and in this problem, we will prove two fundamental properties of a tree, and derive two definitions of a tree we learned from the note based on these properties. Let's start with the properties:

(a) Prove that any pair of vertices in a tree are connected by exactly one (simple) path.

(b) Prove that adding any edge (not already in the graph) between two vertices of a tree creates a simple cycle.

Now you will show that if a graph satisfies this property then it must be a tree:

(c) Prove that if the graph has no simple cycles and has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

Solution:

(a) Pick any pair of vertices x, y. We know there is a path between them since the graph is connected. We will prove that this path is unique by contradiction: Suppose there are two distinct paths from x to y. At some point (say at vertex a) the paths must diverge, and at some point (say at vertex b) they must reconnect. So by following the first path from a to b and the second path in reverse from b to a we get a cycle. This gives the necessary contradiction.

(b) Pick any pair of vertices x, y not connected by an edge. We prove that adding the edge $\{x,y\}$ will create a simple cycle. From part (a), we know that there is a unique path between x and y. Therefore, adding the edge $\{x,y\}$ creates a simple cycle obtained by following the path from x to y, then following the edge $\{x,y\}$ from y back to x.

(c) Assume we have a graph with no simple cycles, but adding any edge will create a simple cycle. We will show that the graph is a tree. We know the graph is acyclic because it has no simple cycles. To show the graph is connected, we prove that any pair of vertices x, y are connected by a path. We consider two cases: If $\{x,y\}$ is an edge, then clearly there is a path from x to y. Otherwise, if $\{x,y\}$ is not an edge, then by assumption, adding the edge $\{x,y\}$ will create a simple cycle. This means there is a simple path from x to y obtained by removing the edge $\{x,y\}$ from this cycle. Therefore, we conclude the graph is a tree.

2 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0, 1\}^n$ (recall that $\{0, 1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

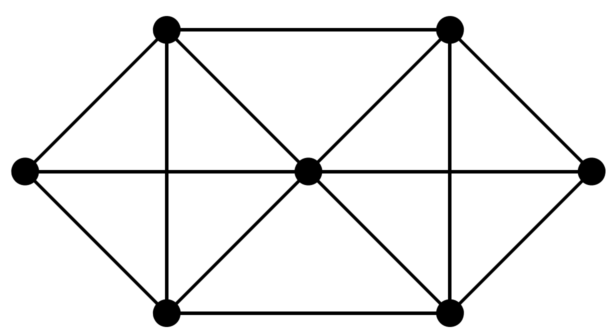
(b) Show that the vertices of an n-dimensional hypercube can be colored using 2 colors so that no pair of adjacent vertices have the same color. This is equivalent to showing that a hypercube is *bipartite*: the vertices can be partitioned into two groups (according to color) so that every edge goes between the two groups.

Solution:

(a) The three hypercubes are a line, a square, and a cube, respectively.

(b) Consider the vertices with an even number of 0 bits and the vertices with an odd number of 0 bits. Each vertex with an even number of 0 bits is adjacent only to vertices with an odd number of 0 bits, since each edge represents a single bit change (either a 0 bit is added by flipping a 1 bit, or a 0 bit is removed by flipping a 0 bit). By using color 0 to color the vertices with an even number of 0 bits and using color 1 to color vertices with an odd number of 0 bits, no two adjacent vertices will share a color.

3 Eulerian Tour and Eulerian Walk



1. Is there an Eulerian tour in the graph above?

2. Is there an Eulerian walk in the graph above?

3. What is the condition that there is an Eulerian walk in an undirected graph?

Solution:

1. No. Two vertices have odd degree.
2. Yes. One of the two vertices with odd degree must be the first vertex, and the other one must be the last vertex.
3. An undirected graph has an Eulerian walk if and only if it is connected (except for isolated vertices) and at most two vertices have odd degree.

   Note: There is no graph with only one odd degree vertex.

4 Hamiltonian Tour in a Hypercube

An alternative type of tour to an Eulerian Tour in graph is a Hamiltonian Tour: a tour that visits every vertex exactly once. Prove or disprove that the hypercube contains a Hamiltonian cycle, for hypercubes of dimension $n \geq 2$.
Hint: When proceeding by induction, a good place to start is writing out what this tour would look like in a 3-dimensional hypercube when starting from the 000 vertex, and using the recursive definition of an n-dimensional hypercube.

Solution:

Going off the hint, we get the following hamiltonian tours:

- $n = 2 : 00, 01, 11, 10$.
- $n = 3 : 000, 001, 011, 010, 110, 111, 101, 100$ [Take the $n = 2$ tour in the 0-subcube (vertices with a 0 in front), move to the 1-subcube (vertices with 1 in front), then take the tour backwards. We know 100 connects to 000 to complete the tour.]

What we've done here is essentially take the tour in the 0-subcube (except for the last edge), transition into the 1-subcube, take the exact same tour in the 1-subcube but backwards, and end at the starting vertex. We can use analogous reasoning to prove this claim with induction on a strengthened inductive hypothesis:
Stronger Inductive Claim: There exists a tour in an n-dimensional hypercube that uses the edge: $(0^n,\ 10^{n-1})$.
Base Case: $n = 2$, the hypercube is just a four cycle, which is a cycle that contains the edge $(00, 10)$ as required.

Inductive Hypothesis: We assume the claim holds for dimension n.

Inductive Step: The recursive definition of an $n+1$ dimensional hypercube is to take two n dimensional hypercubes, relabel each vertex x in one "subcube" as $0x$ and relabel each vertex in the other "subcube" as $1x$ and add edges $(0x, 1x)$ for each $x \in \{0, 1\}^n$.

Use the inductive hypothesis to form seperate tours of each subcube which in the 0th subcube contains the edge $(00^n, 010^{n-1})$ and the 1th subcube contains $(10^n, 110^{n-1})$. We remove these edges then add the edges between the subcubes; $(00^n, 10^n)$ and $(010^{n-1}, 110^{n-1})$.

Notice we do not change the degrees of any node in this swap thus the degree of all the nodes is two.

Moreover, the tour is connected as one can reach every node from all zeros in the first cube using the inductive tour, and in the second cube using the edge to the second cube and the rest of the inductive tour.