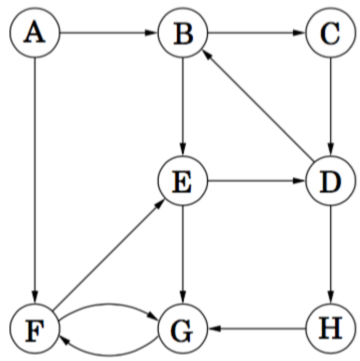
CS 70 Summer 2017 — Discrete Mathematics and Probability Theory — Lu, Moulos, and Tang — DIS 2B

1 Graph Basics

In the first few parts, you will be answering questions on the following graph G.



(a) What are the vertex and edge sets V and E for graph G?

(b) Which vertex has the highest in-degree? Which vertex has the lowest in-degree? Which vertices have the same in-degree and out-degree?

(c) What are the paths from vertex B to F, assuming no vertex is visited twice? Which one is the shortest path?

(d) Which of the following are cycles in G?

i. $\{(B,C),(C,D),(D,B)\}$
ii. $\{(F,G),(G,F)\}$
iii. $\{(A,B),(B,C),(C,D),(D,B)\}$
iv. $\{(B,C),(C,D),(D,H),(H,G),(G,F),(F,E),(E,D),(D,B)\}$

(e) Which of the following are walks in G?

i. $\{(E,G)\}$
ii. $\{(E,G),(G,F)\}$
iii. $\{(F,G),(G,F)\}$
iv. $\{(A,B),(B,C),(C,D)\}$
v. $\{(E,G),(G,F),(F,G),(G,F)\}$
vi. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

(f) Which of the following are tours in G?

i. $\{(E,G)\}$

ii. $\{(E,G),(G,F)\}$

iii. $\{(F,G),(G,F)\}$

iv. $\{(E,D),(D,B),(B,E),(E,D),(D,H),(H,G),(G,F)\}$

In the following three parts, let's consider a general undirected graph G with n vertices ($n \geq 3$).

(g) True/False: If each vertex of G has degree at most 1, then G does not have a cycle.

(h) True/False: If each vertex of G has degree at least 2, then G has a cycle.

(i) True/False: If each vertex of G has degree at most 2, then G is not connected.

Solution:

(a) A graph is specified as an ordered pair $G = (V,E)$, where V is the vertex set and E is the edge set.

$$V = \{A,B,C,D,E,F,G,H\},$$
$$E = \{(A,B),(A,F),(B,C),(B,E),(C,D),(D,B),(D,H),(E,D),(E,G),(F,E),(F,G),(G,F),(H,G)\}.$$

(b) G has the highest in-degree (3). A has the lowest in-degree (0).

$\{B,C,D,E,F,H\}$ all have the same in-degree and out-degree. H and C has in-degree (out-degree) equal to 1 and the other four have in-degree (out-degree) equal to 2.

(c) There are three paths:
$\{(B,C),(C,D),(D,H),(H,G),(G,F)\}$ (length = 5)
$\{(B,E),(E,D),(D,H),(H,G),(G,F)\}$ (length = 5)
$\{(B,E),(E,G),(G,F)\}$ (length = 3)

The last one listed above has the shortest path.

(d) A cycle should be a path that starts and ends at the same point, so iii is not a cycle. In addition, all the vertices $\{v_1,\ldots,v_n\}$ in the cycle $\{(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n)\}$ should be distinct, so iv is not a cycle. The correct answers are i and ii.

(e) All of them. A walk can end on the same vertex on which it begins or on a different vertex. A walk can travel over any edge and any vertex any number of times.

(f) iii. A tour is a walk which starts and ends at the same vertex.

(g) True. In a cycle, every vertex has degree at least 2.

(h) True. Consider starting a walk at some vertex v_0, and at each step, walking along a previously untraversed edge, stopping when we first visit some vertex w for the second time. If such a process succeeds, then the part of our walk from the first time we visited w until the second time is a cycle, so it remains only to argue this process succeeds. Each time we take a step from some vertex v, since we are not stopping, we must have visited that vertex exactly once and not yet left. It follows that we have used at most one edge incident with v (either we started at v, or we took an edge into v). Since v has degree 2, there must be another edge leaving v for us to take.

(i) False. For example, a 3-cycle (triangle) is connected and every vertex has degree 2.

2 Build-Up Error?

What is wrong with the following "proof"?

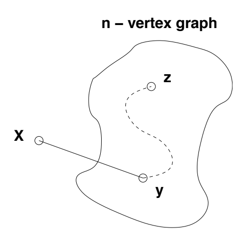
False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$.

Solution:

The mistake is in the argument that *"every $(n+1)$-vertex graph with minimum degree 1 can be obtained from an n-vertex graph with minimum degree 1 by adding 1 more vertex"*. Instead of starting by considering an arbitrary $(n+1)$-vertex graph, this proof only considers an $(n+1)$-vertex graph that you can make by starting with an n-vertex graph with minimum degree 1. As a counterexample, consider a graph on four vertices $V = \{1,2,3,4\}$ with two edges $E = \{\{1,2\},\{3,4\}\}$. Every vertex in this graph has degree 1, but there is no way to build this 4-vertex graph from a 3-vertex graph with minimum degree 1.

More generally, this is an example of *build-up error* in proof by induction. Usually this arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others, such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a *"shrink down, grow back"* process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds.

Let's see what would have happened if we'd tried to prove the claim above by this method. In the inductive step, we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$- vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh! The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck —— and properly so, since the claim is false!

3 Bipartite Graph

Consider an undirected bipartite graph with two disjoint sets L, R. Prove that a graph is bipartite if and only if it no cycles of odd length.

Solution:

Begin by proving the forward direction: an undirected bipartite graph has no cycles of odd length.

Let us start traveling the cycle from a node n_0 in L. Since each edge in the graph connects a vertex in L to one in R, the 1st edge in the set connects our start node n_0 to the a node n_1 in R. The 2nd edge in the cycle must connect n_1 to a node n_2 in L. Continuing on, the $(2k+1)$-th edge connects node n_{2k} in L to node n_{2k+1} in R, and the $2k$-th edge connects node n_{2k-1} in R to node n_{2k} in L. Since only even numbered edges connect to vertices in L, and we started our cycle in L, the cycle must end with an even number of edges.

Prove the reverse direction: A undirected graph with no cycles of odd length is bipartite.

Take some vertex v. Add all vertices where the shortest path to v is odd, to R. Add all vertices where the shortest path to v is even, to L. If any of the vertices in $u_1, u_2 \in R$ are connected, then we have a cycle of odd length: (v, u_1) (odd), (u_1, u_2) (odd), and (u_2, v) (odd). This means no two vertices in R are connected. We can pick any vertex in R to repeat, and we have that L, R are disjoint.