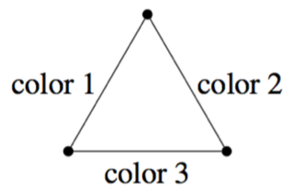
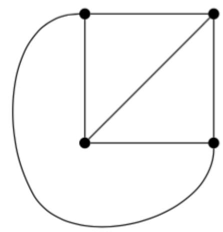
CS 70 Summer 2018 — Discrete Mathematics and Probability Theory — Sinho Chewi and Vrettos Moulos — DIS 2B

1 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.





(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1,2,3$ for colors. A figure is shown on the right.)

(b) How many colors are required to edge color a 3-dimensional hypercube?

(c) Prove that any graph with maximum degree d can be edge colored with $2d-1$ colors.

(d) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Solution:

(a) Three color a triangle. Add the fourth vertex, notice that each edge has a different color available from the set of three colors.

(b) 3. Recall that edges connect vertices that differ in a dimension. And each vertex is incident to exactly one edge for each dimension. Thus, the entire set of edges for a specific dimension can be colored with a single color.

(c) By induction on the number of edges. We will use a set of $2d-1$ colors. Remove an edge and $2d-1$ color the remaining graph from our set. This can be done by the induction hypothesis as the remaining graph's degree is no bigger than d and the graph has fewer edges. The edge is incident to two vertices each of which is incident to at most $d-1$ other edges, and thus at most $2(d-1) = 2d-2$ colors are unavailable for edge e. Thus, we can color edge e without any conflicts.

(d) By induction on the number of vertices. Base case is a single vertex, which has no edges to color, and thus can be colored with 0 colors. Remove the degree 1 vertex, v. Color the remaining tree with d colors. Note that vertex v's neighboring vertices has degree at most $d-1$ without the edge to v and thus its incident edges use at most $d-1$ colors. Thus, there is a color available for coloring the edge incident to this vertex.

2 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices, such that no 2 vertices in the same set have an edge between them. Consider an undirected bipartite graph with two disjoint sets L, R. Prove that a graph is bipartite if and only if it has no tours of odd length.

Solution:

Begin by proving the forward direction: an undirected bipartite graph has no tours of odd length.

Let us start traveling the tour from a node n_0 in L. Since each edge in the graph connects a vertex in L to one in R, the 1st edge in the tour connects our start node n_0 to a node n_1 in R. The 2nd edge in the tour must connect n_1 to a node n_2 in L. Continuing on, the $(2k+1)$-th edge connects node n_{2k} in L to node n_{2k+1} in R, and the $2k$-th edge connects node n_{2k-1} in R to node n_{2k} in L. Since only even numbered edges connect to vertices in L, and we started our tour in L, the tour must end with an even number of edges.

Prove the reverse direction: A undirected graph with no tours of odd length is bipartite.

Take some vertex v. Add all vertices where the shortest path to v is odd, to R. Add all vertices where the shortest path to v is even, to L. If any of the vertices in $u_1, u_2 \in R$ are connected, then we have a tour of odd length formed by appending: the shortest path between v and u_1 (odd), the edge (u_1, u_2) (odd), and the shortest path between u_2 and v (odd). This means no two vertices in R are connected. Similarly, if any two vertices $v_1, v_2 \in L$ are connected, we get a tour of odd length by appending: the shortest path between v and v_1 (even), the edge (v_1, v_2) (odd), and the shortest path between v_2 and v (even). This means no two vertices in L are connected either. If there are other connected componenets, we can proceed by choosing a new vertex in each component and repeating this process. Then we will have disjoint L, R which include all vertices.

3 Modular Arithmetic Equations

Solve the following equations for x and y modulo the indicated modulus, or show that no solution exists. Show your work.

(a) $9x \equiv 1 \pmod{11}$.

(b) $10x + 23 \equiv 3 \pmod{31}$.

(c) $3x + 15 \equiv 4 \pmod{21}$.

(d) The system of simultaneous equations $3x+2y \equiv 0 \pmod 7$ and $2x+y \equiv 4 \pmod 7$.

Solution:

(a) Multiply both sides by $9^{-1} \equiv 5 \pmod{11}$ to get $x \equiv 5 \pmod{11}$.

(b) Subtract 23 from both sides, then multiply both sides by $10^{-1} = -3 \pmod{31}$ to find $x \equiv (-20) \cdot (-3) \equiv 60 \equiv 29 \pmod{31}$.

(c) Subtract 15 from both sides to get $3x \equiv 10 \pmod{21}$. Now note that this implies $3x \equiv 1 \pmod 3$, since 3 divides 21, and the latter equation has no solution, so the former cannot either.

We are using the fact that if $d \mid m$, then $x \equiv y \pmod m$ implies $x \equiv y \pmod d$ (but not necessarily the other way around). To see this, if $x \equiv y \pmod m$, then $m \mid x-y$ (by definition) and so $d \mid x-y$, and hence $x \equiv y \pmod d$.

(d) First, subtract the first equation from double the second equation to get $2(2x+y)-(3x+2y) \equiv x \equiv 1 \pmod 7$; now plug in to the second equation to get $2+y \equiv 4 \pmod 7$, so the system has the solution $x \equiv 1 \pmod 7$, $y \equiv 2 \pmod 7$.