1 Trees

Recall that a *tree* is a connected acyclic graph (graph without cycles). In the note, we presented a few other definitions of a tree, and in this problem, we will prove two fundamental properties of a tree, and derive two definitions of a tree we learned from the note based on these properties. Let's start with the properties:

(a) Prove that any pair of vertices in a tree are connected by exactly one (simple) path.

(b) Prove that adding any edge (not already in the graph) between two vertices of a tree creates a simple cycle.

Now you will show that if a graph satisfies this property then it must be a tree:

(c) Prove that if the graph has no simple cycles and has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

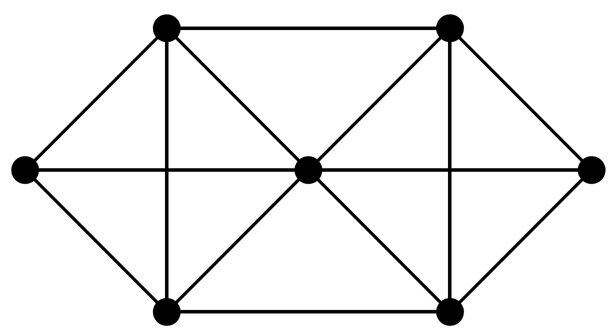
2 Hypercubes

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that the vertices of an n-dimensional hypercube can be colored using 2 colors so that no pair of adjacent vertices have the same color. This is equivalent to showing that a hypercube is *bipartite*: the vertices can be partitioned into two groups (according to color) so that every edge goes between the two groups.

3 Eulerian Tour and Eulerian Walk



1. Is there an Eulerian tour in the graph above?
2. Is there an Eulerian walk in the graph above?
3. What is the condition that there is an Eulerian walk in an undirected graph?

4 Hamiltonian Tour in a Hypercube

An alternative type of tour to an Eulerian Tour in graph is a Hamiltonian Tour: a tour that visits every vertex exactly once. Prove or disprove that the hypercube contains a Hamiltonian cycle, for hypercubes of dimension $n \geq 2$.
Hint: When proceeding by induction, a good place to start is writing out what this tour would

look like in a 3-dimensional hypercube when starting from the 000 vertex, and using the recursive definition of an n-dimensional hypercube.