1 Trees

Recall that a *tree* is a connected acyclic graph (graph without cycles). In the note, we presented a few other definitions of a tree, and in this problem, we will prove two fundamental properties of a tree, and derive two definitions of a tree we learn from lecture note based on these properties. Let's start with the properties:

(a) Prove that any pair of vertices in a tree are connected by exactly one (simple) path.

(b) Prove that adding any edge between two vertices of a tree creates a simple cycle.

Now you will show that if a graph satisfies either of these two properties then it must be a tree:

(c) Prove that if every pair of vertices in a graph are connected by exactly one simple path, then the graph must be a tree.

(d) Prove that if the graph has no simple cycles and has the property that the addition of any single edge (not already in the graph) will create a simple cycle, then the graph is a tree.

Solution:

(a) Pick any pair of vertices x, y. We know there is a path between them since the graph is connected. We will prove that this path is unique by contradiction: Suppose there are two distinct paths from x to y. At some point (say at vertex a) the paths must diverge, and at some point (say at vertex b) they must reconnect. So by following the first path from a to b and the second path in reverse from b to a we get a cycle. This gives the necessary contradiction.

(b) Pick any pair of vertices x, y not connected by an edge. We prove that adding the edge $\{x,y\}$ will create a simple cycle. From part (a), we know that there is a unique path between x and y. Therefore, adding the edge $\{x,y\}$ creates a simple cycle obtained by following the path from x to y, then following the edge $\{x,y\}$ from y back to x.

(c) Assume we have a graph with the property that there is a unique simple path between every pair of vertices. We will show that the graph is a tree, namely, it is connected and acyclic. First, the graph is connected because every pair of vertices is connected by a path. Moreover, the graph is acyclic because there is a unique path between every pair of vertices. More explicitly, if the graph has a cycle, then for any two vertices x, y in the cycle there are at least two simple paths between them (obtained by going from x to y through the right or left half of the cycle), contradicting the uniqueness of the path. Therefore, we conclude the graph is a tree.

(d) Assume we have a graph with no simple cycles, but adding any edge will create a simple cycle. We will show that the graph is a tree. We know the graph is acyclic because it has no simple cycles. To show the graph is connected, we prove that any pair of vertices x, y are connected by a path. We consider two cases: If $\{x,y\}$ is an edge, then clearly there is a path from x to y. Otherwise, if $\{x,y\}$ is not an edge, then by assumption, adding the edge $\{x,y\}$ will create a simple cycle. This means there is a simple path from x to y obtained by removing the edge $\{x,y\}$ from this cycle. Therefore, we conclude the graph is a tree.

2 Planarity

Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Prove that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Solution:

Assume G is planar. Take 5 vertices, they cannot form K_5, so some pair v_1, v_2 have no edge between them. The remaining five vertices of G cannot form K_5 either, so there is a second pair v_3, v_4 that have no edge between them. Now consider v_1, v_2 and any other three vertices v_5, v_6, v_7. Since v_1v_2 is not an edge, by property T it must be that v_1v and v_2v where $v \in \{v_5, v_6, v_7\}$ are edges. Similarly for v_3, v_4, v_3v and v_4v where $v \in \{v_5, v_6, v_7\}$ are edges. So now any three vertices in $\{v_1, v_2, v_3, v_4\}$ on one side and $\{v_5, v_6, v_7\}$ on the other form an instance of $K_{3,3}$. Contradiction.

The above shows that any graph with 7 vertices and property T is non-planar. Any graph with > 7 vertices and property T will also be non-planar because it will contain a subgraph with 7 vertices and property T.

3 Graph Coloring

Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

Solution:

The natural way to try to prove this theorem is to use induction on k. Unfortunately, this approach leads to disaster. It is not that it is impossible, just that it is extremely painful and would ruin your week if you tried it on an exam. When you encounter such a disaster when using induction on graphs, it is usually best to change what you are inducting on. In graphs, typical good choices for the induction parameter are n, the number of nodes, or e, the number of edges.

We use induction on the number of vertices in the graph, which we denote by n. Let $P(n)$ be the proposition that an n-vertex graph with maximum degree at most k is $(k+1)$-colorable.

Base Case $n = 1$: A 1-vertex graph has maximum degree 0 and is 1-colorable, so $P(1)$ is true.

Inductive Step: Now assume that $P(n)$ is true, and let G be an $(n+1)$-vertex graph with maximum degree at most k. Remove a vertex v (and all edges incident to it), leaving an n-vertex subgraph,H. The maximum degree of H is at most k, and so H is $(k+1)$-colorable by our assumption $P(n)$.

Now add back vertex v. We can assign v a color (from the set of $k+1$ colors) that is different from all its adjacent vertices, since there are at most k vertices adjacent to v and so at least one of the $k+1$ colors is still available. Therefore, G is $(k+1)$-colorable. This completes the inductive step, and the theorem follows by induction.

4 Hamiltonian Tour in a Hypercube

An alternative type of tour to an Eulerian Tour in graph is a Rudrata Tour: a tour that visits every vertex exactly once. Prove or disprove that the hypercube contains a Rudrata cycle, for hypercubes of dimension $n \geq 2$.

Solution:

We will strengthen the inductive hypothesis.

Stronger Inductive Claim: There exists a tour in an n-dimensional hypercube that uses the edge: $(0^n, 10^{n-1})$.

Base Case: $n = 2$, the hypercube is just a four cycle, which is a cycle that contains the edge $(00, 10)$ as required.

Inductive Hypothesis: We assume the claim holds for dimension n.

Inductive Step: The recursive definition of an $n+1$ dimensional hypercube is to take two n dimensional hypercubes, relabel each vertex x in one "subcube" as $0x$ and relabel each vertex in the other "subcube" as $1x$ and add edges $(0x, 1x)$ for each $x \in \{0,1\}^n$.

Use the inductive hypothesis to form seperate tours of each subcube which in the 0th subcube contains the edge $(00^{n-1}, 010^{n-2})$ and the 1th subcube contains $(10^{n-1}, 110^{n-1})$. We remove these edges then add the edges between the subcubes; $(00^{n-1}, 10^{n-1})$ and $(010^{n-2}, 110^{n-2})$.

Notice we do not change the degrees of any node in this swap thus the degree of all the nodes is two.

Moreover, the tour is connected as one can reach every node from all zeros in the first cube using the inductive tour, and in the second cube using the edge to the second cube and the rest of the inductive tour.