1 Cube Dual

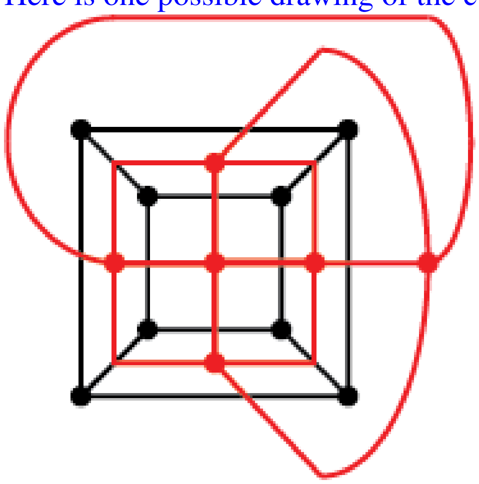
We define a graph G by letting the vertices be the corners of a cube and having edges connecting adjacent corners. Define the *dual* of a planar graph G to be a graph G', constructed by replacing each face in G with a vertex, and an edge between every vertex in G' if the respective faces are adjacent in G.

(a) Draw a planar representation of G and the corresponding dual graph. Is the dual graph planar? (Hint: think about the act of drawing the dual)

(b) Is G' bipartite?

Solution:

(a) Here is one possible drawing of the cube (in black) with its dual (in red):



As seen in the drawing, the dual is indeed planar.

(b) From the drawing, G' is not bipartite. This is a reminder that connecting the middle of every face on a cube does not result in another cube, which would be bipartite!

2 True or False

(a) Any pair of vertices in a tree are connected by exactly one path.

(b) Adding an edge between two vertices of a tree creates a new cycle.

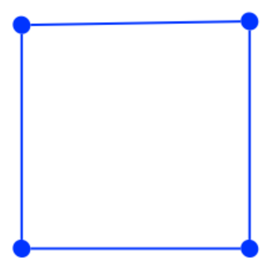
(c) Adding an edge in a connected graph creates exactly one new cycle.

Solution:

(a) **True.**
Pick any pair of vertices x, y. We know there is a path between them since the graph is connected. We will prove that this path is unique by contradiction: Suppose there are two distinct paths from x to y. At some point (say at vertex a) the paths must diverge, and at some point (say at vertex b) they must reconnect. So by following the first path from a to b and the second path in reverse from b to a we get a cycle. This gives the necessary contradiction.

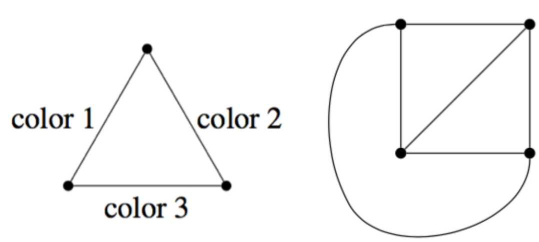
(b) **True.**
Pick any pair of vertices x, y not connected by an edge. We prove that adding the edge $\{x,y\}$ will create a cycle. From part (a), we know that there is a unique path between x and y. Therefore, adding the edge $\{x,y\}$ creates a cycle obtained by following the path from x to y, then following the edge $\{x,y\}$ from y back to x.

(c) **False.**
In the following graph adding an edge creates two cycles.



3 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



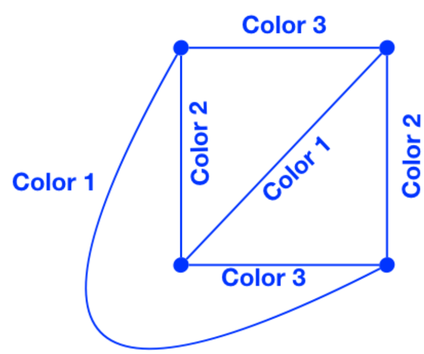
(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1,2,3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d-1$ colors.

(c) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Solution:

(a) Three color a triangle. Now add the fourth vertex notice, call it vertex u. For any edge, say $\{u, v\}$ from this fourth vertex u, observe that the vertex v has two edges from before and hence there a third color available for the edge $\{u, v\}$.



(b) We will use induction on the number of edges n in the graph to prove the statement: If a graph G has $n \geq 0$ edges and the maximum degree of any vertex is d, then G can be colored with $2d-1$ colors.
Base case (n = 0). If there are no edges in the graph, then there is nothing to be colored and the statement holds trivially.
Inductive hypothesis. Suppose for $n = k \geq 0$, the statement holds.
Inductive step. Consider a graph G with $n = k+1$ edges. Remove an edge of your choice, say e from G. Note that in the resulting graph the maximum degree of any vertes is $d' \leq d$. By the inductive hypothesis, we can color this graph using $2d'-1$ colors and hence with $2d-1$ colors too. The removed edge is incident to two vertices each of which is incident to at most $d-1$ other edges, and thus at most $2(d-1) = 2d-2$ colors are unavailable for edge e. Thus, we can color edge e without any conflicts. This proves the statement for $n = k+1$ and hence by induction we get that the statement holds for all $n \geq 0$.

(c) We will use induction on the number of vertices n in the tree to prove the statement: For a tree with $n \geq 1$ vertices, if the maximum degree of any vertex is d, then the tree can be colored with d colors.
Base case (n=1). If there is only one vertex, then there are no edges to color, and thus can be colored with 0 colors.
Inductive hypothesis. Suppose the statement holds for $n = k \geq 1$.
Inductive Step. Remove any leaf v of your choice from the tree. We can then color the remaining tree with d colors by the inductive hypothesis. For any neighboring vertex u of vertex v,

the degree of u is at most $d-1$ since we removed the edge $\{u,v\}$ along with the vertex v. Thus its incident edges use at most $d-1$ colors and there is a color available for coloring the edge {u,v}. This completes the inductive step and by induction we have that the statement holds for all $n \geq 1$.