CS 70 Discrete Mathematics and Probability Theory
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1 Clothes and Stuff

(a) Say we've decided to do the whole capsule wardrobe thing and we now have only 5 different items of clothing that we wear (jeans, tees, shoes, jackets, and floppy hats, etc.). We have 3 variations on each of the items, and we wear one of each item every day. How many different outfits can we make?

(b) It turns out 3 floppy hats really isn't enough of a selection, so we've bought 11 more, and we now have 14 floppy hats. Now how many outfits can we make?

(c) If we own k different items of clothing, with n_1 varitions of the first item, n_2 variations of the second, n_3 of the third, and so on, how many outfits can we make?

(d) We love our floppy hats so much that we've decided to also use them as wall art, so we're picking 4 of our 14 hats to hang in a row on the wall. How many such arrangements could we make? (Order matters.)

(e) Ok, now we're packing for vacation to Iceland, and we only have space for 4 of our 14 floppy hats. How many sets of 4 could we bring? (Yeah, yeah, we knew you were going to use that notation. Now tell us the number as a function of d, your answer from the previous part.)

(f) Ok, turns out the check-in person for our flight to Iceland is being *very* unreasonable about the luggage weight restrictions, and we're going to have to leave some hats behind. Despite our best intentions, and having packed only 4 hats, we actually bought 18 additional floppy hats at the airport (6 in burgundy, 6 in forest green, and 6 in classic black). We'll keep our 4 hats that we brought from home, but we'll have to return all but 6 of the airport hats. How many color configurations can there be for the 6 airport hats that we keep?

Solution:

(a) 3^5

(b) $14 \cdot 3^4$

(c) $n_1 \cdot n_2 \cdot n_3 \cdots n_k$

(d) $14!/10!$

(e) $\binom{14}{4}$ or written as a function of the previous part, $d/4!$

(f) Within each color category, the hats are indistinguishable, so we will use the stars-and-bars method. Specifically, consider the six choices you have to make to be six “stars”, and you must place your stars in one of three color categories. Since there are three categories, we need two “bars” (dividers) to separate the categories. In total, we have $6+2=8$ stars and bars, and so there are $\binom{8}{6}$ ways to choose the positions of the stars (which corresponds to $\binom{8}{6}$ ways to choose our floppy hats). Equivalently, there are $\binom{8}{2}$ ways to choose the positions of the bars.

But let’s be serious, you should just keep the black ones – so much more versatile.

2 Fruits

Suppose you want to buy n fruits, and you can buy 0 or more of any type. In how how many ways can you do that if:

(a) There are apples and oranges at the market.

(b) There are apples, oranges, and bananas at the market.

(c) There are k kinds of fruits at the market.

Solution:

This is a classic stars and bars problem.

(a) $n+1$.

(b) $\binom{n+2}{2}$.

(c) $\binom{n+k-1}{k-1}$.

3 Combinatorial Proof I

Prove $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$

Solution:

LHS: first choose a group of r people from n people, then choose k leaders among those r.

RHS: first choose k leaders, then among the rest ($n-k$ people), choose $r-k$ followers to form a group of size r ($r-k+k=r$).

4 Teams and Leaders

Prove the following identities using a combinatorial proof.

1. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$
2. $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

Solution:

1. Imagine you are a teacher picking students to be on a team for some competition. You have $2n$ students, n of whom are boys and the other n are girls.

   RHS: This is simply the number of ways you can pick n students to be on the team.
   LHS: We begin by noticing that $\binom{n}{k}^2 = \binom{n}{k} \cdot \binom{n}{n-k}$. This product gives us the number of ways of picking k girls and $n-k$ boys to be on the team. We add up all the products involving anywhere from 0 girls all the way to n girls. This gives us the total number of ways to pick a team of n students.

2. Imagine the same scenario as part (a) except now you have to choose a female team leader amongst the n students on the team.

   RHS: This is the number of ways of picking the team leader multiplied with the number of ways of picking the rest of the team from the remaining students. The product gives the total number of teams with a female leader.
   LHS: We begin similarly by noticing that $k \cdot \binom{n}{k}^2 = k \cdot \binom{n}{k} \cdot \binom{n}{n-k}$. Here as before we are picking k girls and $n-k$ boys to be on the team. However amongst the k girls on the team, we choose one of them to be the team leader. We add up all the products involving anywhere from 0 girls all the way to n girls. This gives us the total number of ways to pick a team of n students with a female leader.