

1 Modular Arithmetic Equations

Solve the following equations for x and y modulo the indicated modulus, or show that no solution exists. Show your work.

(a) $9x \equiv 1 \pmod{11}$.

(b) $10x + 23 \equiv 3 \pmod{31}$.

(c) $3x + 15 \equiv 4 \pmod{21}$.

(d) The system of simultaneous equations $3x + 2y \equiv 0 \pmod{7}$ and $2x + y \equiv 4 \pmod{7}$.

2 Bijections

Let n be an odd number. Let $f(x)$ be a function from $\{0, 1, \dots, n-1\}$ to $\{0, 1, \dots, n-1\}$. In each of these cases say whether or not $f(x)$ is necessarily a bijection. Justify your answer (either prove $f(x)$ is a bijection or give a counterexample).

(a) $f(x) = 2x \pmod{n}$.

(b) $f(x) = 5x \pmod{n}$.

(c) n is prime and

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x^{-1} \pmod{n} & \text{if } x \neq 0. \end{cases}$$

(d) n is prime and $f(x) = x^2 \pmod{n}$.

3 Baby Fermat

Assume that a does have a multiplicative inverse mod m . Let us prove that its multiplicative inverse can be written as $a^k \pmod{m}$ for some $k \geq 0$.

(a) Consider the sequence $a, a^2, a^3, \dots \pmod{m}$. Prove that this sequence has repetitions.

(b) Assuming that $a^i \equiv a^j \pmod{m}$, where $i > j$, what can you say about $a^{i-j} \pmod{m}$?

(c) Prove that the multiplicative inverse can be written as $a^k \pmod{m}$. What is k in terms of i and j ?

4 Combining Moduli

Suppose we wish to work modulo $n = 40$. Note that $40 = 5 \times 8$, with $\gcd(5, 8) = 1$. We will show that in many ways working modulo 40 is the same as working modulo 5 and modulo 8, in the sense that instead of writing down $c \pmod{40}$, we can just write down $c \pmod{5}$ and $c \pmod{8}$.

(a) What is $8 \pmod{5}$ and $8 \pmod{8}$? Find a number $a \pmod{40}$ such that $a \equiv 1 \pmod{5}$ and $a \equiv 0 \pmod{8}$.

- (b) Now find a number $b \pmod{40}$ such that $b \equiv 0 \pmod{5}$ and $b \equiv 1 \pmod{8}$.
- (c) Now suppose you wish to find a number $c \pmod{40}$ such that $c \equiv 2 \pmod{5}$ and $c \equiv 5 \pmod{8}$. Find c by expressing it in terms of a and b .
- (d) Repeat to find a number $d \pmod{40}$ such that $d \equiv 3 \pmod{5}$ and $d \equiv 4 \pmod{8}$.
- (e) Compute $c \times d \pmod{40}$. Is it true that $c \times d \equiv 2 \times 3 \pmod{5}$, and $c \times d \equiv 5 \times 4 \pmod{8}$?