1 Flippin' Coins

Suppose we have a biased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 3/4$ and probability of tails $\mathbb{P}[T] = 1/4$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

Solution:

(a) $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

(b) An event must be a subset of Ω, meaning that it must consist of possible outcomes.

- No
- Yes
- Yes
- Yes
- Yes

(c) $\{(T,H,H),(T,H,T),(T,T,H)\}$

(d) $\{(T,H,H),(H,H,T),(H,T,H),(T,T,T)\}$

(e) $\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{64}$

(f) $\omega \in \{(H,H,T),(H,T,H),(T,H,H)\}$. The probability = $3 \cdot \frac{9}{64} = \frac{27}{64}$.

2 Venn Diagram

Out of 1000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.

(a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.

(b) What is the probability that the student belongs to a club?

(c) What is the probability that the student works part time?

(d) What is the probability that the student belongs to a club AND works part time?

(e) What is the probability that the student belongs to a club OR works part time?

Solution:

(a) The sample space will be illustrated by a Venn diagram.

(b) $\mathbb{P}[C] = \frac{|C|}{|\Omega|} = \frac{400}{1000} = \frac{2}{5}$.

(c) $\mathbb{P}[P] = \frac{|P|}{|\Omega|} = \frac{500}{1000} = \frac{1}{2}$.

(d) $\mathbb{P}[P \cap C] = \frac{|P \cap C|}{|\Omega|} = \frac{50}{1000} = \frac{1}{20}$.

(e) $\mathbb{P}[P \cup C] = \mathbb{P}[P] + \mathbb{P}[C] - \mathbb{P}[P \cap C] = \frac{1}{2} + \frac{2}{5} - \frac{1}{20} = \frac{17}{20}$.

3 Probability Practice

(a) If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?

(b) A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?

(c) If five numbers are selected at random from the set $\{1,2,3,\ldots,20\}$, what is the probability that their minimum is larger than 5? (A number can be chosen more than once.)

Solution:

(a) $18!5!/22! = 1/1463$. The 18! comes from 18 "units": 3 physics books, 8 engineering books, 6 biology books and 1 block of math books. The 5! comes from number of ways to arrange the 5 math books within the same block. 22! is just the total number of ways to arrange the books.

(b) $(2/3)^8 = 256/6561$. This is just by independence.

(c) $(15/20)^5 = 243/1024$. For a single number, we can choose $\{6,7,\ldots,20\}$, so 15 valid outcomes out of 20 total outcomes. Then apply independence as in part (b).

4 Birthdays

Suppose you record the birthdays of a large group of people, one at a time until you have found a match, i.e., a birthday that has already been recorded. (Assume there are 365 days in a year.)

(a) What is the probability that after the first 3 people's birthdays are recorded, no match has occurred (i.e. each person has a unique birthday)?

(b) What is the probability that the first 3 people all share the same birthday?

(c) What is the probability that it takes more than 20 people for a match to occur?

(d) What is the probability that it takes exactly 20 people for a match to occur?

(e) Suppose instead that you record the birthdays of a large group of people, one at a time, until you have found a person whose birthday matches your own birthday. What is the probability that it takes exactly 20 people for this to occur?

Solution:

(a) $\frac{364}{365} \cdot \frac{363}{365}$.

(b) $\left(\frac{1}{365}\right)^2$.

(c)

$$\begin{aligned}\mathbb{P}[\text{it takes more than 20 people}] &= \mathbb{P}[\text{20 people don't have the same birthday}] \\ &= \frac{365!/(365-20)!}{365^{20}} = \boxed{\frac{\mathbf{365!}}{\mathbf{345!365^{20}}}} \approx .589.\end{aligned}$$

Another explanation that does not use counting:

The first person can have any birthday. The second person must have a different birthday from the first person, which occurs with probability $364/365$. The third person must have a different birthday from the first two people, which occurs with probability $363/365$. Generalizing, the ith person must have a different birthday from the first $i-1$ people, which occurs with probability $(365-(i-1))/365$. Hence,

$$\begin{aligned}\mathbb{P}[\text{it takes more than 20 people}] &= \frac{365-19}{365} \times \frac{365-18}{365} \times \cdots \times \frac{363}{365} \times \frac{364}{365} \\ &\approx .589.\end{aligned}$$

(d) The probability that it takes exactly 20 people is the probability that the first 19 people have different birthdays **and** the 20th person shares a birthday with one of the first 19 people.

How many total ways can the birthdays be chosen for 20 people? 365^{20}.

How many ways can the birthdays be chosen so the first 19 have different birthdays and the 20^{th} person shares a birthday with the first 19? Well, the first person has 365 choices, the second has 364 choices left, and so on until the nineteenth person has $(365-19+1) = 347$ choices left. Then, the 20^{th} person has 19 choices for his birthday. So in total, there are $365 \cdot 364 \cdots 348 \cdot 347 \cdot 19 = (365!/346!) \cdot 19$ ways of getting what we want. So

$$\mathbb{P}[\text{it takes exactly 20 people}] = \frac{365 \cdot 364 \cdots 348 \cdot 347 \cdot 19}{365^{20}} = \boxed{\frac{\mathbf{365! \cdot 19}}{\mathbf{346!365^{20}}}} \approx .032.$$

Another explanation that does not use counting:

As before, the ith person must have a different birthday from the first $i-1$ people, with probability $(365-(i-1))/365$, for $i = 1, \ldots, 19$. The 20th person must share a birthday with one of the first 19 people (who all have distinct birthdays), so the probabiltiy is $19/365$. Hence,

$$\begin{aligned}\mathbb{P}[\text{it takes exactly 20 people}] &= \frac{19}{365} \times \frac{365-18}{365} \times \cdots \times \frac{363}{365} \times \frac{364}{365} \\ &\approx .032.\end{aligned}$$

(e) The probability that it takes exactly 20 people is the probability that the first 19 people don't have your birthday and the 20th person has your birthday.

Similar to the last problem, there are 364 choices for the first person's birthday to be different than yours, 364 for the second person, and so on until the nineteenth person has 364 choices.

Then, the 20^{th} person has exactly 1 choice to have your birthday. So the total number of ways to get what we want is $364^{19} \cdot 1$. There are 365^{20} possibilities total. So

$$\mathbb{P}[\text{it takes exactly 20 people}] = \boxed{\frac{\mathbf{364^{19}}}{\mathbf{365^{20}}}} \approx .0026.$$

Another explanation that does not use counting:

Each of the 19 people who do not share your birthday do so with probability $364/365$, and the last person must share your birthday with probability $1/365$. Hence,

$$\begin{aligned}\mathbb{P}[\text{it takes exactly 20 people}] &= \frac{364}{365} \times \frac{364}{365} \times \cdots \times \frac{364}{365} \times \frac{1}{365} \\ &= \frac{364^{19} \times 1}{365^{20}} \\ &\approx 0.0026.\end{aligned}$$