CS 70 Discrete Mathematics and Probability Theory
Summer 2018 Sinho Chewi and Vrettos Moulos **DIS 3D**

1 Countability Basics

1. Is $f : \mathbb{N} \to \mathbb{N}$, defined by $f(n) = n^2$ an injection (one-to-one)? Briefly justify.
2. Is $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3 + 1$ a surjection (onto)? Briefly justify.

Solution:

1. Yes. One way to illustrate is by drawing the one-to-one mapping from n to n^2. More formally we can show that the preimage is unique by showing that $m \neq n \implies f(m) \neq f(n)$.

   We'll do proof by contraposition. $f(m) = f(n) \implies m = n$.

   $$f(m) = f(n) \implies m^2 = n^2 \implies m^2 - n^2 = 0 \implies (m-n)(m+n) = 0 \implies m = \pm n$$

   Since n can't be negative, we have an injection.

2. Yes. For any value of y, there always exists a corresponding input x. If $y = x^3 + 1$, we know that $x = \sqrt[3]{y-1}$. Thus for any value of y, there exists this value of x which maps to it.

2 Count It!

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) $\mathbb{N}$, the set of all natural numbers.

(b) $\mathbb{Z}$, the set of all integers.

(c) $\mathbb{Q}$, the set of all rational numbers.

(d) $\mathbb{R}$, the set of all real numbers.

(e) The integers which divide 8.

(f) The integers which 8 divides.

(g) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(h) Numbers that are the roots of nonzero polynomials with integer coefficients.

Solution:

(a) Countable and infinite. See Lecture Note 10.

(b) Countable and infinite. See Lecture Note 10.

(c) Countable and infinite. See Lecture Note 10.

(d) Uncountable. This can be proved using a diagonalization argument, as shown in class. See Lecture Note 10.

(e) Finite. They are $\{-8,-4,-2,-1,1,2,4,8\}$.

(f) Countably infinite. We know that there exists a bijective function $f : \mathbb{N} \to \mathbb{Z}$. Then function $g(n) = 8f(n)$ is a bijective mapping from $\mathbb{N}$ to integers which 8 divides.

(g) Uncountably infinite. We use the Cantor's Diagonalization Proof:

Let $\mathscr{F}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. We can represent a function $f \in \mathscr{F}$ as an infinite sequence $(f(0), f(1), \cdots)$, where the i-th element is $f(i)$. Suppose towards a contradiction that there is a bijection from $\mathbb{N}$ to $\mathscr{F}$:

$$\begin{aligned} 0 &\longleftrightarrow (f_0(0), f_0(1), f_0(2), f_0(3), \ldots) \\ 1 &\longleftrightarrow (f_1(0), f_1(1), f_1(2), f_1(3), \ldots) \\ 2 &\longleftrightarrow (f_2(0), f_2(1), f_2(2), f_2(3), \ldots) \\ 3 &\longleftrightarrow (f_3(0), f_3(1), f_3(2), f_3(3), \ldots) \\ &\vdots \end{aligned}$$

Consider the function $g : \mathbb{N} \to \mathbb{N}$ where $g(i) = f_i(i) + 1$ for $i \in \mathbb{N}$. We claim that the function g is not in our finite list of functions. Suppose for contradiction that it did, and that it was the n-th function $f_n(\cdot)$ in the list, i.e., $g(\cdot) = f_n(\cdot)$. However, $f_n(\cdot)$ and $g(\cdot)$ differ in the n-th number, i.e. $f_n(n) \neq g(n)$, because by our construction $g(n) = f_n(n) + 1$ (Contradiction!).

(h) Countably infinite. The total number of programs is countably infinite, since each can be viewed as a string of characters (so for example if we assume each character is one of the 256 possible values, then each program can be viewed as number in base 256, and we know these numbers are countably infinite). So the number of halting programs, which is a subset of all programs, can be either finite or countably infinite. But there are an infinite number of halting programs, for example for each number i the program that just prints i is different for each i. So the total number of halting programs is countably infinite.

3 Countability Practice

(a) Prove or disprove: The set of increasing functions $f : \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \geq f(y)$) is countable.

(b) Prove or disprove: The set of decreasing functions $f : \mathbb{N} \to \mathbb{N}$ (i.e., if $x \geq y$, then $f(x) \leq f(y)$) is countable.

(c) Is a set of disks in $\mathbb{R}^2$ such that no two disks overlap necessarily countable or possibly uncountable? [A disk is a region in the plane of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq r^2\}$, for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$.]

(d) Is a set of circles in $\mathbb{R}^2$ such that no two circles overlap necessarily countable or possibly uncountable? [*Hint*: A circle is a subset of the plane of the form $\{(x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 = r^2\}$ for some $x_0, y_0, r \in \mathbb{R}$, $r > 0$. The difference between a circle and a disk is that a disk contains all of the points in its interior, whereas a circle does not.]

Solution:

(a) Suppose that there is a bijection between $\mathbb{N}$ and the set of all increasing functions $\mathbb{N} \to \mathbb{N}$:

$$\begin{aligned}
0 &\mapsto (f_0(0), f_0(1), f_0(2), \ldots) \\
1 &\mapsto (f_1(0), f_1(1), f_1(2), \ldots) \\
2 &\mapsto (f_2(0), f_2(1), f_2(2), \ldots) \\
&\vdots
\end{aligned}$$

We will use a diagonalization argument to prove that there is a function f which is not in the above list. Define

$$f(n) = 1 + \sum_{i=1}^{n} f_i(n).$$

First, we will show that f is increasing. Indeed, if $m \leq n$, then

$$f(m) = 1 + \sum_{i=1}^{m} f_i(m) \leq 1 + \sum_{i=1}^{n} f_i(m) \leq 1 + \sum_{i=1}^{n} f_i(n) = f(n).$$

The first inequality is because each function is non-negative; the second inequality is because the f_i are increasing.

To show that f is not in the list, note that

$$f(n) = 1 + \sum_{i=1}^{n} f_i(n) \geq 1 + f_n(n) > f_n(n).$$

Since $f(n) > f_n(n)$ for each $n \in \mathbb{N}$, f cannot be any of the functions in the list. Therefore, the set of increasing functions $f : \mathbb{N} \to \mathbb{N}$ is uncountable.

(b) Given any function that begins with $f(0) = n$, consider the number of indices in which the function decreases in output: the set of i such that $f(i) < f(i-1)$. There are only at most n such indices because eventually, the function will hit $f(n) = 0$ for which every subsequent input will output 1. We can set a bijection for any function with $f(0) = n$ to a "word" of indices at which the function decreases. Therefore, the set of decreasing functions $\mathbb{N} \to \mathbb{N}$ has the same cardinality as the set of finite bit strings from a countably infinite alphabet, which is countable. Therefore, the set of all decreasing functions is countable.

(c) Countable. Each disk must contain at least one rational point (an (x, y)-coordinate where $x, y \in \mathbb{Q}$) in its interior, and due to the fact that no two disks overlap, the cardinality of the set of disks can be no larger than the cardinality of $\mathbb{Q} \times \mathbb{Q}$, which we know to be countable.

(d) Possibly uncountable. Consider the circles $C_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r\}$ for each $r \in \mathbb{R}$. For $r_1 \neq r_2$, C_{r_1} and C_{r_2} do not overlap, and there are uncountably many of these circles (one for each real number).