1 Balls and Bins

You have n empty bins and you throw balls into them one by one randomly. A collision is when a ball is thrown into a bin which already has another ball.

(a) What is the probability that the first ball thrown will cause the first collision?

(b) What is the probability that the second ball thrown will cause the first collision?

(c) What is the probability that, given the first two balls are not in collision, the third ball thrown will cause the first collision?

(d) What is the probability that the third ball thrown will cause the first collision?

(e) What is the probability that, given the first $m-1$ balls are not in collision, the m^{th} ball thrown will cause the first collision?

(f) What is the probability that the m^{th} ball thrown will cause the first collision?

Solution:

(a) 0

(b) $\frac{1}{n}$

(c) $\frac{2}{n}$

(d) $\mathbb{P}(\text{Ball 3 collides} \mid \text{Balls 1, 2 do not collide}) \cdot \mathbb{P}(\text{Balls 1, 2 do not collide})$, which is $\frac{2}{n} \cdot \frac{n-1}{n}$

(e) $\frac{m-1}{n}$

(f) Similar to (d), $\frac{m-1}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-m+2}{n} = \frac{m-1}{n} \cdot \prod_{i=0}^{m-2} \frac{n-i}{n}$

2 To Be Fair

Suppose you have a biased coin with $\mathbb{P}(\text{heads}) \neq 0.5$. How could you use this coin to simulate a fair coin? (*Hint*: Think about pairs of tosses.)

Solution:

Let H be the event that the coin flip comes up heads and T be the event that the coin flip comes up tails. We do not know the probability of getting heads since we've been told the coin is biased, but let's denote it by the variable p (i.e., $p = \mathbb{P}(H)$).

We conduct a mini experiment in which we flip the same coin twice. Let HT be the event that we get heads and then tails in two consecutive flips. Similarly, let TH be the event that we get tails and then heads in two consecutive flips. Also, we'll let HH be the event that we get two heads in a row, and TT be the event that we get two tails in a row. Our sample space for the experiment is then $\Omega = \{HH, HT, TH, TT\}$.

Since we know $\mathbb{P}(H) = p$, we can write the probabilities of the events in our sample space as follows:

$$\mathbb{P}(HH) = \mathbb{P}(\text{first toss } H)\,\mathbb{P}(\text{second toss } H \mid \text{first toss } H)$$

Because the nature of coins tells us that the second toss doesn't remember the first toss so

$$\mathbb{P}(HH) = \mathbb{P}(H)\,\mathbb{P}(H) = p^2.$$

Similarly,

$$\begin{aligned}
\mathbb{P}(HT) &= \mathbb{P}(\text{first toss } H)\,\mathbb{P}(\text{second toss } T \mid \text{first toss } H) = \mathbb{P}(H)(1-\mathbb{P}(H)) = p(1-p), \\
\mathbb{P}(TH) &= \mathbb{P}(\text{first toss } T)\,\mathbb{P}(\text{second toss } H \mid \text{first toss } T) = (1-\mathbb{P}(H))\,\mathbb{P}(H) = p(1-p), \\
\mathbb{P}(TT) &= \mathbb{P}(\text{first toss } T)\,\mathbb{P}(\text{second toss } T \mid \text{first toss } T) = (1-\mathbb{P}(H))(1-\mathbb{P}(H)) = (1-p)^2.
\end{aligned}$$

We notice that the probability $\mathbb{P}(HT)$ and the probability $\mathbb{P}(TH)$ are equal, i.e., they are both $p(1-p)$. By symmetry, these two probabilities must be the same since the coin doesn't know what it came up before. Since these two are the same, we can simply condition on the fact that something was returned to get that the resulting simulated coin toss is fair.

Therefore, we can simulate a fair coin using the following process. We toss the coin twice. If the outcome turns out to be heads both times or tails both times, we throw away the result and repeat the whole process again. Otherwise, if the outcome is HT, we return "heads", and if the outcome is TH, we return "tails".

How do we know this procedure will return a result at all? How do we know that this can't go on forever? We are not yet in a position to prove this rigorously because we haven't built the tools yet. But intuitively, this would require getting an infinite sequence of HH's or TT's. Why would the coin always agree with itself? This seems like it defies the nature of coin tossing: that the coin doesn't know what it did before.

3 Poisoned Smarties

Supposed there are 3 men who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the men, produces considerably more Smarties than his competitors and has a commanding 45% of the market share. Yousef See, who inherited his riches, lags behind Burr and produces 35% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 20%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through his investigations, the inspector found that one Smarty out of every 100 at Kelly's factory was poisonous. At See's factory, 1.5% of Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.02.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

Solution:

(a) Let S be the event that a smarty is safe to eat.
Let BK be the event that a smarty is from Burr Kelly's factory.
Let YS be the event that a smarty is from Yousef See's factory.
Finally, let SF be the event that a smarty is from Stan Furd's factory.

$$\begin{aligned}\mathbb{P}(S) &= \mathbb{P}(BK)P(S \mid BK) + \mathbb{P}(YS)\,\mathbb{P}(S \mid YS) + \mathbb{P}(SF)\,\mathbb{P}(S \mid SF)\\ &= (0.45)(0.99) + (0.35)(0.985) + (0.2)(0.98) = 0.98625.\end{aligned}$$

Therefore the probability that a Smarty is safe to eat is about 0.98625.

(b) Let P be the event that a smarty is poisonous.

$$\begin{aligned}\mathbb{P}(P \mid \neg BK) &= \mathbb{P}(YS \mid \neg BK)\,\mathbb{P}(P \mid YS) + \mathbb{P}(SF \mid \neg BK)\,\mathbb{P}(P \mid SF)\\ &= \frac{\mathbb{P}(YS)}{\mathbb{P}(\neg BK)}\mathbb{P}(P \mid YS) + \frac{\mathbb{P}(SF)}{\mathbb{P}(\neg BK)}\mathbb{P}(P \mid SF)\\ &= \frac{0.35}{0.55} \cdot 0.015 + \frac{0.2}{0.55} \cdot 0.02 = 0.0168.\end{aligned}$$

(c)

$$\mathbb{P}(SF \mid P) = \frac{\mathbb{P}(P \mid SF)\,\mathbb{P}(SF)}{\mathbb{P}(P)}$$

In the first part we calculate the probability that any random Smarty was safe to eat. We can use that since $\mathbb{P}(P) = 1 - \mathbb{P}(S)$. Therefore the solution becomes:

$$\begin{aligned}\mathbb{P}(SF \mid P) &= \frac{\mathbb{P}(P \mid SF)\,\mathbb{P}(SF)}{1-\mathbb{P}(S)} \\ &= \frac{(0.02)(0.2)}{(1-0.98625)} = 0.29.\end{aligned}$$

4 Rain and Wind

The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in EECS 70, you are curious to play around with these numbers. Find the probability that:

(a) A given day is both windy and rainy.

(b) A given day is rainy.

(c) For a given pair of days, exactly one of the two days is rainy.

(d) A given day that is non-rainy is also non-windy.

Solution:

(a) Let R be the event that it rains on a given day and W be the event that a given day is windy. We are given $\mathbb{P}(R \mid W) = 0.3$, $\mathbb{P}(R \mid W^c) = 0.8$ and $\mathbb{P}(W) = 0.2$. Then probability that a given day is both rainy and windy is $\mathbb{P}(R \cap W) = \mathbb{P}(R \mid W)\,\mathbb{P}(W) = 0.3 \times 0.2 = 0.06$.

(b) Probability that it rains on a given day is $\mathbb{P}(R) = \mathbb{P}(R \mid W)\,\mathbb{P}(W) + \mathbb{P}(R \mid W^c)\,\mathbb{P}(W^c) = 0.3 \times 0.2 + 0.8 \times 0.8 = 0.7$.

(c) Let R_1 and R_2 be the events that it rained on day 1 and day 2 respectively. Since the days are independent, $\mathbb{P}(R_1) = \mathbb{P}(R_2) = \mathbb{P}(R)$. The required probability is $\mathbb{P}(R_1)\,\mathbb{P}(R_2^c) + \mathbb{P}(R_1^c)\,\mathbb{P}(R_2) = 2 \cdot 0.7 \cdot 0.3 = 0.42$.

(d) Probability that a non-rainy day is non-windy is

$$\mathbb{P}(W^c \mid R^c) = \frac{\mathbb{P}(W^c \cap R^c)}{\mathbb{P}(R^c)} = \frac{\mathbb{P}(R^c \mid W^c)(W^c)}{\mathbb{P}(R^c)} = \frac{0.2 \times 0.8}{0.3} = \frac{8}{15}.$$