1 Anagrams

An anagram of a word is any re-ordering of the letters of the word, in any order. It does not have to be an English word.

(a) How many different anagrams are there of COVERAGE?

(b) How many different anagrams are there of COVFEFE?

(c) How many different anagrams are there of COVFEFES that contain EECS?

Solution:

(a)

$$\frac{8!}{2!}$$

(b)

$$\frac{7!}{2!2!}$$

(c)

$$\frac{5!}{2!}$$

2 Counting Mappings

(a) A mapping $f : X \to Y$ is a function from X to Y, which assigns a unique element $f(x) \in Y$ for each $x \in X$. How many unique mappings are there between $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, m\}$?

(b) A mapping f is *injective* if for all $x_1, x_2 \in X$, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. How many injective mappings are there between $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, m\}$?

(c) Now suppose that we allow a mapping to be multi-valued, i.e. for each x, $f(x)$ is a subset of Y. How many unique multi-valued mappings are there between $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, m\}$?

Solution:

(a) For each $x \in X$ there are m choices, and there are n elements in X. Thus, we are making a series of n choices each of which gives us m options, for a total of m^n different possible mappings.

(b) If $m < n$, there cannot be any injective mappings, which follows from a simple proof by contradiction. Otherwise, there are m choices for $f(1)$, and given this choice, there are $m-1$ choices for $f(2)$, and so on. Thus, the total number of injective mappings is $m(m-1)(m-2)...(m-n+1) = \frac{m!}{(m-n)!}$.

(c) For each $x \in X$ there are 2^m choices for what subset of Y it can map to. So by similar logic to the first part, the total number of possible multi-valued mappings is $(2^m)^n = 2^{mn}$.

3 Clothes and Stuff

(a) Say we've decided to do the whole capsule wardrobe thing and we now have only 5 different items of clothing that we wear (jeans, tees, shoes, jackets, and floppy hats, etc.). We have 3 variations on each of the items, and we wear one of each item every day. How many different outfits can we make?

(b) It turns out 3 floppy hats really isn't enough of a selection, so we've bought 11 more, and we now have 14 floppy hats. Now how many outfits can we make?

(c) If we own k different items of clothing, with n_1 varitions of the first item, n_2 variations of the second, n_3 of the third, and so on, how many outfits can we make?

(d) We love our floppy hats so much that we've decided to also use them as wall art, so we're picking 4 of our 14 hats to hang in a row on the wall. How many such arrangements could we make? (Order matters.)

(e) Ok, now we're packing for vacation to Iceland, and we only have space for 4 of our 14 floppy hats. How many sets of 4 could we bring? (Yeah, yeah, we knew you were going to use that notation. Now tell us the number as a function of d, your answer from the previous part.)

(f) Ok, turns out the check-in person for our flight to Iceland is being *very* unreasonable about the luggage weight restrictions, and we're going to have to leave some hats behind. Despite our best intentions, and having packed only 4 hats, we actually bought 18 additional floppy hats at the airport (6 in burgundy, 6 in forest green, and 6 in classic black). We'll keep our 4 hats that we brought from home, but we'll have to return all but 6 of the airport hats. How many color configurations can there be for the 6 airport hats that we keep?

Solution:

(a) 3^5

(b) $14 \cdot 3^4$

(c) $n_1 \cdot n_2 \cdot n_3 \cdots n_k$

(d) $14!/10!$

(e) $\binom{14}{4}$ or written as a function of the previous part, $d/4!$

(f) Within each color category, the hats are indistinguishable, so we will use the stars-and-bars method. Specifically, consider the six choices you have to make to be six "stars", and you must place your stars in one of three color categories. Since there are three categories, we need two "bars" (dividers) to separate the categories. In total, we have $6+2=8$ stars and bars, and so there are $\binom{8}{6}$ ways to choose the positions of the stars (which corresponds to $\binom{8}{6}$ ways to choose our floppy hats). Equivalently, there are $\binom{8}{2}$ ways to choose the positions of the bars.

But let's be serious, you should just keep the black ones – so much more versatile.

4 Combinatorial Proof IX

Prove that for $0 < n < k$, $\binom{n}{k} = \sum_{i=0}^{k} \binom{n-i-1}{k-i}$.

Solution:

The left hand side of the equation is just the typical way of counting the number of bit strings of length n that have exactly k ones.

For the right hand side, we first look at a single element of the summation. This counts the number of bit strings of length n that have exactly k ones and that have their first zero at position $i+1$. To see why this is, consider that in order for a bit string's first zero to come at position $i+1$, the first i digits have to all be ones. Thus, there are $i+1$ digits that need to be fixed. Once we've dealt with those, there are $n-i-1$ positions left that are not fixed, of which $k-i$ need to be ones (since we already used up i ones in the first $i+1$ digits). Hence, there are $\binom{n-i-1}{k-i}$ ways to complete the string, as claimed.

Since $n > k$, we know that every bit string must have at least one zero–and since there can only be k ones, we know that the first zero can appear no later that position $k+1$. Thus, by summing over i from 0 to k, we cover each of the bit strings covered by the left hand side exactly once, and so the two sides must be equal.