1 Bayes Rule – Man Speaks Truth

(a) A man speaks the truth 3 out of 4 times. He flips a biased coin that comes up Heads $1/3$ of the time and reports that it is Heads. What is the probability it is Heads?

(b) A man speaks the truth 3 out of 4 times. He rolls a fair 6-sided die. When you ask him if the die came up with a 6, he answers "yes". What is the probability it is really 6?

Solution:

(a) Let E denote the event the man reports heads, S_1 be the event that the coin comes up heads, and S_2 be the event that the coin comes up tails.

We have:

$$\mathbb{P}(E \mid S_1) = \frac{3}{4}, \quad \mathbb{P}(E \mid S_2) = \frac{1}{4}, \quad \mathbb{P}(S_1) = \frac{1}{3}, \quad \mathbb{P}(S_2) = \frac{2}{3}.$$

We want to compute $\mathbb{P}(S_1 \mid E)$, and let's do so by applying Bayes Rule.

$$\mathbb{P}(S_1 \mid E) = \frac{\mathbb{P}(S_1 \cap E)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E \mid S_1)\mathbb{P}(S_1)}{\mathbb{P}(E \mid S_1)\mathbb{P}(S_1) + \mathbb{P}(E \mid S_2)\mathbb{P}(S_2)} = \frac{(3/4)\cdot(1/3)}{(3/4)\cdot(1/3)+(1/4)\cdot(2/3)}$$
$$= \frac{3}{5}.$$

(b) Let E denote the event that the man says "yes" to your question, S_1 be the event that the dice comes up 6, and S_2 be the event that the dice comes up not 6.

We have:

$$\mathbb{P}(E \mid S_1) = \frac{3}{4}, \quad \mathbb{P}(E \mid S_2) = \frac{1}{4}, \quad \mathbb{P}(S_1) = \frac{1}{6}, \quad \mathbb{P}(S_2) = \frac{5}{6}.$$

We want to compute $\mathbb{P}(S_1 \mid E)$, and let's do so by applying Bayes Rule.

$$\mathbb{P}(S_1 \mid E) = \frac{\mathbb{P}(S_1 \cap E)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E \mid S_1)\mathbb{P}(S_1)}{\mathbb{P}(E \mid S_1)\mathbb{P}(S_1) + \mathbb{P}(E \mid S_2)\mathbb{P}(S_2)} = \frac{(3/4)\cdot(1/6)}{(3/4)\cdot(1/6)+(1/4)\cdot(5/6)}$$
$$= \frac{3}{8}.$$

2 Disease Diagnosis

You have a high fever and go to the doctor to identify the cause. 1% of the people have H1N1, 10% of the people have the flu, and 89% have neither. Assume that no person has both. Suppose that 100% of the H1N1 people have a high fever, 30% of the flu people have a high fever, and 2% of the people who have neither, have a high fever. Is it more likely that you have H1N1, the flu, or neither?

Solution:

Let A be the event that the patient has H1N1, B be the event that the patient has flu, and C be the event that the patient has neither. The event of having a fever is D. We want to compare $\mathbb{P}(A \mid D)$, $\mathbb{P}(B \mid D)$, and $\mathbb{P}(C \mid D)$. We find each value using Bayes rule.

$$\begin{aligned}\mathbb{P}(A \mid D) &= \frac{\mathbb{P}(D \mid A)\mathbb{P}(A)}{\mathbb{P}(D \mid A)\mathbb{P}(A) + \mathbb{P}(D \mid B)\mathbb{P}(B) + \mathbb{P}(D \mid C)\mathbb{P}(C)} \\ &= \frac{1 \times 0.01}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} \\ &= 0.173\end{aligned} \tag{1}$$

$$\begin{aligned}\mathbb{P}(B \mid D) &= \frac{\mathbb{P}(D \mid B)\mathbb{P}(B)}{\mathbb{P}(D \mid A)\mathbb{P}(A) + \mathbb{P}(D \mid B)\mathbb{P}(B) + \mathbb{P}(D \mid C)\mathbb{P}(C)} \\ &= \frac{0.1 \times 0.3}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} \\ &= 0.519\end{aligned} \tag{2}$$

$$\begin{aligned}\mathbb{P}(C \mid D) &= \frac{\mathbb{P}(D \mid C)\mathbb{P}(C)}{\mathbb{P}(D \mid A)\mathbb{P}(A) + \mathbb{P}(D \mid B)\mathbb{P}(B) + \mathbb{P}(D \mid C)\mathbb{P}(C)} \\ &= \frac{0.89 \times 0.02}{1 \times 0.01 + 0.1 \times 0.3 + 0.89 \times 0.02} \\ &= 0.308\end{aligned} \tag{3}$$

So flu is the most likely.

3 Pairwise Independence

The events A_1, A_2, A_3 are *pairwise independent* if, for all $i \neq j$, A_i is independent of A_j. However, pairwise independence is a weaker statement than *mutual independence*, which requires the additional condition that $\mathbb{P}(A_1, A_2, A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$.

Try to construct an example where three events are pairwise independent but not mutually independent.

Here is one potential starting point: Let A_1, A_2 be the respective results of flipping two fair coins. Can you come up with an event A_3 that works?

Solution:

A_1: the first result is Head; A_2: the second result is Head; A_3: both results are the same.

4 Balls and Bins

Throw n balls into n bins.

(a) What is the probability that the first bin is empty?

(b) What is the probability that the first k bins are empty?

(c) Give an upper bound on the probability that at least k bins are empty.

(d) What is the probability that the second bin is empty given that the first one is empty?

(e) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

(f) Are the events that "the first bin is empty" and "the second bin is empty" independent?

Solution:

(a) $\left(\frac{n-1}{n}\right)^n$.

(b) $\left(\frac{n-k}{n}\right)^n$.

(c) We use the union bound. Let A be the event that at least k bins are empty. Notice that there are $m = \binom{n}{k}$ sets of k bins out of the total n bins. Then

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} \mathbb{P}(A_i)$$

where A_i is the event that the ith set of k bins is empty. We know the probability of the first k bins being empty from part (b), and this is true for any set of k bins, so

$$\mathbb{P}(A_i) = \left(\frac{n-k}{n}\right)^n.$$

Then,

$$\mathbb{P}(A) \leq m \cdot \left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n.$$

(d) Using probability rules:

$$\begin{aligned}\mathbb{P}[\text{2nd bin empty} \mid \text{1st bin empty}] &= \frac{\mathbb{P}[\text{2nd bin empty} \cap \text{1st bin empty}]}{\mathbb{P}[\text{1st bin empty}]} \\ &= \frac{(n-2)^n/n^n}{(n-1)^n/n^n} \\ &= \left(\frac{n-2}{n-1}\right)^n\end{aligned} \tag{4}$$

Alternate solution:
We know bin 1 is empty, so each ball that we throw can land in one of the remaining $n-1$ bins. We want the probability that bin 2 is empty, which means that each ball cannot land in bin 2 either, leaving $n-2$ bins. Thus for each ball, the probability that bin 2 is empty given that bin 1 is empty is $(n-2)/(n-1)$. For n total balls, this probability is $[(n-2)/(n-1)]^n$.

(e) They are dependent. Knowing the latter means the former happens with probability 1.

(f) In part (c) we calculated the probability that the second bin is empty given that the first bin is empty: $[(n-2)/(n-1)]^n$. The probability that the second bin is empty (without any prior information) is $[(n-1)/n]^n$. Since these probabilities are not equal, the events are dependent.