CS 70 Summer 2017 Discrete Mathematics and Probability Theory Hongling Lu, Vrettos Moulos, and Allen Tang DIS 4C

1 Head Count

Consider flipping a fair coin twice.

(a) What is the sample space Ω generated from these flips?

(b) Define a random variable X to be the number of heads. What is the distribution of X?

(c) Define a random variable Y to be 1 if $\omega = (H,T)$ and 0 otherwise. What is the distribution of Y?

(d) Define a third random variable $Z = X + Y$. What is the distribution of Z?

Solution:

(a) $\{(T,T),(H,T),(T,H),(H,H)\}$.

(b)

$$X = \begin{cases} 0 & \text{w.p.} \quad .25 \\ 1 & \text{w.p.} \quad .5 \\ 2 & \text{w.p.} \quad .25 \end{cases}$$

(c)

$$Y = \begin{cases} 0 & \text{w.p.} \quad .75 \\ 1 & \text{w.p.} \quad .25 \end{cases}$$

(d) Let's determine the values Z can take and the corresponding probabilities:

- $Z = 0$: $\mathbb{P}(Z=0) = \mathbb{P}(X=0 \cap Y=0) = \mathbb{P}(X=0) \cdot \mathbb{P}(Y=0 \mid X=0) = .25 \cdot 1 = .25$
- $Z = 1$:

$$\begin{aligned} \mathbb{P}(Z=1) &= \mathbb{P}(X=0 \cap Y=1) + \mathbb{P}(X=1 \cap Y=0) \\ &= \mathbb{P}(X=0) \cdot \mathbb{P}(Y=1 \mid X=0) + \mathbb{P}(X=1) \cdot \mathbb{P}(Y=0 \mid X=1) \qquad (1) \\ &= .25 \cdot 0 + .5 \cdot .5 = .25 \end{aligned}$$

- $Z = 2$:

$$\begin{aligned} \mathbb{P}(Z=2) &= \mathbb{P}(X=1 \cap Y=1) + \mathbb{P}(X=2 \cap Y=0) \\ &= \mathbb{P}(X=1) \cdot \mathbb{P}(Y=1 \mid X=1) + \mathbb{P}(X=2) \cdot \mathbb{P}(Y=0 \mid X=2) \qquad (2) \\ &= .5 \cdot 5 + .25 \cdot 1 = .5 \end{aligned}$$

- $Z=3$: $\mathbb{P}(Z=3)=\mathbb{P}(X=2\cap Y=1)=\mathbb{P}(X=2)\cdot\mathbb{P}(Y=1\mid X=2)=.25\cdot 0=0$

$$Z=\begin{cases}0 & \text{w.p.} \quad .25\\ 1 & \text{w.p.} \quad .25\\ 2 & \text{w.p.} \quad .5\end{cases}$$

2 Head Count II

Now consider a new coin with $\mathbb{P}(\text{Heads})=2/5$. We'll flip the coin 20 times.

(a) As before, define X to be the number of heads. What is $\mathbb{P}(X=7)$?

(b) What is $\mathbb{P}(X\geq 1)$?

(c) What is $\mathbb{P}(12\leq X\leq 14)$?

Solution:

(a) X is a binomially distributed random variable.

$$\mathbb{P}(X=7)=\binom{20}{7}\left(\frac{2}{5}\right)^{7}\left(\frac{3}{5}\right)^{13}.$$

(b)

$$\mathbb{P}(X\geq 1)=1-\mathbb{P}(X=0)=1-\left(\frac{3}{5}\right)^{20}.$$

(c)

$$\mathbb{P}(12\leq X\leq 14)=\binom{20}{12}\left(\frac{2}{5}\right)^{12}\left(\frac{3}{5}\right)^{8}+\binom{20}{13}\left(\frac{2}{5}\right)^{13}\left(\frac{3}{5}\right)^{7}+\binom{20}{14}\left(\frac{2}{5}\right)^{14}\left(\frac{3}{5}\right)^{6}.$$

3 Telebears

Lydia has just started her Telebears appointment. She needs to register for a marine science class and CS 70. There are no waitlists, and she can attempt to enroll once per day in either class or both. The Telebears system is strange and picky, so the probability of enrolling in the marine science class is p_1 and the probability of enrolling in CS 70 is p_2. The probabilities are independent. Let M be the number of days it takes to enroll in the marine science class, and C be the number of days it takes to enroll in CS 70.

(a) What distribution do M and C follow? Are M and C independent?

(b) For some integer $k \geq 1$, what is $\mathbb{P}[C \geq k]$?

(c) For some integer $k \geq 1$, what is the probability that she is enrolled in both classes before day k?

Solution:

(a) $M \sim \text{Geom}(p_1)$, $C \sim \text{Geom}(p_2)$. Yes they are independent.

(b) We are looking for the probability that it takes at least k days to enroll in CS 70. Using the geometric distribution, this is $(1-p_2)^{k-1}$.

(c) Use independence. Let X be the number of days before she is enrolled in both.

$$\begin{aligned}\mathbb{P}[X < k] &= \mathbb{P}[M < k]\mathbb{P}[C < k] = (1-\mathbb{P}[M \geq k])(1-\mathbb{P}[C \geq k]) \\ &= (1-(1-p_1)^{k-1})(1-(1-p_2)^{k-1})\end{aligned}$$

4 Fishy Computations

Use the Poisson distribution to answer these questions:

(a) Suppose that on average, a fisherman catches 20 salmon per week. What is the probability that he will catch exactly 7 salmon this week?

(b) Suppose that on average, you go to Fisherman's Wharf twice a year. What is the probability that you will go at most once in 2018?

(c) Suppose that in March, on average, there are 5.7 boats that sail in Laguna Beach per day. What is the probability there will be *at least* 3 boats sailing throughout the *next two days* in Laguna?

Solution:

(a) $X \sim \text{Poiss}(20)$.

$$\mathbb{P}[X = 7] = \frac{20^7}{7!}e^{-20} \approx 5.23 \cdot 10^{-4}.$$

(b) $X \sim \text{Poiss}(2)$.

$$\mathbb{P}[X \leq 1] = \frac{2^0}{0!}e^{-2} + \frac{2^1}{1!}e^{-2} \approx 0.41.$$

(c) Let Y be the number of boats that sail in the next two days. We can approximate Y as a Poisson distribution $Y \sim \text{Poiss}(\lambda = 11.4)$, where λ is the average number of boats that sail over two days. Now, we compute

$$\begin{aligned}
\mathbb{P}[Y \geq 3] &= 1 - \mathbb{P}[Y < 3] \\
&= 1 - \mathbb{P}[Y = 0 \cup Y = 1 \cup Y = 2] \\
&= 1 - (\mathbb{P}[Y = 0] + \mathbb{P}[Y = 1] + \mathbb{P}[Y = 2]) \\
&= 1 - \left(\frac{11.4^0}{0!} e^{-11.4} + \frac{11.4^1}{1!} e^{-11.4} + \frac{11.4^2}{2!} e^{-11.4} \right) \\
&\approx 0.999.
\end{aligned}$$

We can show what we did above formally with the following claim: the sum of two independent Poisson random variables is Poisson. We won't prove this, but from the above, you should intuitively know why this is true. Now, we can model sailing boats on day i as a Poisson distribution $X_i \sim \text{Poiss}(\lambda = 5.7)$. Now, let X_1 be the number of sailing boats on the next day, and X_2 be the number of sailing boats on the day after next. We are interested in $Y = X_1 + X_2$. Thus, we know $Y \sim \text{Poiss}(\lambda = 5.7 + 5.7 = 11.4)$.