1 Proof with Indicators

Let $n \in \mathbb{Z}_+$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and let $A_1, \ldots, A_n$ be events. Prove that $\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \mathbb{P}(A_i \cap A_j) \geq 0$.

Solution:

We write the summation with indicators. Let X_i be the indicator for event A_i, $i = 1, \ldots, n$. Then,

$$\begin{aligned}\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \mathbb{P}(A_i \cap A_j) &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \mathbb{E}[X_i X_j] = \mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j X_i X_j\right] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^n \alpha_i X_i\right)\left(\sum_{j=1}^n \alpha_j X_j\right)\right] = \mathbb{E}\left[\left(\sum_{i=1}^n \alpha_i X_i\right)^2\right] \geq 0.\end{aligned}$$

2 Binomial Conditioning

Let $n \in \mathbb{Z}_+$ and $p, q \in [0,1]$. Let $X \sim \text{Binomial}(n, p)$ and suppose that conditioned on $X = x$, $Y \sim \text{Binomial}(x, q)$. What is the unconditional distribution of Y?

Solution:

Y takes on values in $\{0, \ldots, n\}$. So, let $y \in \{0, \ldots, n\}$.

$$\begin{aligned}\mathbb{P}(Y = y) &= \sum_{x=y}^n \mathbb{P}(X = x)\mathbb{P}(Y = y \mid X = x) = \sum_{x=y}^n \binom{n}{x} p^x (1-p)^{n-x} \binom{x}{y} q^y (1-q)^{x-y} \\ &= \sum_{x=y}^n \frac{n!}{x!(n-x)!} \frac{x!}{y!(x-y)!} p^x (1-p)^{n-x} q^y (1-q)^{x-y} \\ &= \frac{n!}{y!(n-y)!} (pq)^y \sum_{x=y}^n \frac{(n-y)!}{(n-x)!(x-y)!} (p(1-q))^{x-y} (1-p)^{n-x} \\ &= \binom{n}{y} (pq)^y \sum_{x=0}^{n-y} \frac{(n-y)!}{(n-y-x)!x!} (p(1-q))^x (1-p)^{n-y-x} \\ &= \binom{n}{y} (pq)^y \sum_{x=0}^{n-y} \binom{n-y}{x} (p(1-q))^x (1-p)^{n-y-x} = \binom{n}{y} (pq)^y (p(1-q) + (1-p))^{n-y} \\ &= \binom{n}{y} (pq)^y (1-pq)^{n-y}.\end{aligned}$$

We see that $Y \sim \text{Binomial}(n, pq)$.

3 The Memoryless Property Uniquely Characterizes the Geometric Distribution

Let X be a discrete random variable which takes on values on $\mathbb{Z}_+$. Suppose that for all $m,n \in \mathbb{N}$, we have $\mathbb{P}(X > m+n \mid X > n) = \mathbb{P}(X > m)$. Prove that X has the geometric distribution.

Solution:

Notice that

$$\mathbb{P}(X > m+n \mid X > n) = \frac{\mathbb{P}(X > m+n)}{\mathbb{P}(X > n)} = \mathbb{P}(X > m).$$

So, this gives $\mathbb{P}(X > m+n) = \mathbb{P}(X > m)\mathbb{P}(X > n)$. By repeatedly applying this property, we can deduce $\mathbb{P}(X > n) = \mathbb{P}(X > 1 + \cdots + 1) = \mathbb{P}(X > 1)^n$. Let $p := 1 - \mathbb{P}(X > 1)$. We see that $\mathbb{P}(X > n) = (1-p)^n$, which is the tail probability of the geometric distribution, and hence $X \sim \text{Geometric}(p)$.