1  Head Count

Consider a coin with \( \Pr(\text{Heads}) = \frac{2}{5} \). Suppose you flip the coin 20 times, and define \( X \) to be the number of heads.

(a) Name the distribution of \( X \) and what its parameters are.

(b) What is \( \Pr(X = 7) \)?

(c) What is \( \Pr(X \geq 1) \)? Hint: You should be able to do this without a summation.

(d) What is \( \Pr(12 \leq X \leq 14) \)?

**Solution:**

(a) Since we have 20 independent trials, with each trial having a probability \( \frac{2}{5} \) of success, \( X \sim \text{Binomial}(20, \frac{2}{5}) \).

(b) \[
\Pr(X = 7) = \binom{20}{7} \left( \frac{2}{5} \right)^7 \left( \frac{3}{5} \right)^{13}.
\]

(c) \[
\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - \left( \frac{3}{5} \right)^{20}.
\]

(d) \[
\Pr(12 \leq X \leq 14) = \Pr(X = 12) + \Pr(X = 13) + \Pr(X = 14) \\
= \binom{20}{12} \left( \frac{2}{5} \right)^{12} \left( \frac{3}{5} \right)^8 + \binom{20}{13} \left( \frac{2}{5} \right)^{13} \left( \frac{3}{5} \right)^7 + \binom{20}{14} \left( \frac{2}{5} \right)^{14} \left( \frac{3}{5} \right)^6.
\]

2  How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let \( X \) denote the number of queens you draw.

(a) What is \( \Pr(X = 0) \), \( \Pr(X = 1) \), \( \Pr(X = 2) \) and \( \Pr(X = 3) \)?
(b) What do your answers you computed in part a add up to?

(c) Compute $E(X)$ from the definition of expectation.

(d) Let $X_i$ be an indicator random variable that equals 1 if the $i$th card a is queen and 0 otherwise. Are the $X_i$ indicators independent?

Solution:

(a) Calculate each case of $X = 0, 1, 2, 3$:

We must draw 3 non-queen cards in a row, so the probability is

$$P(X = 0) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} = \frac{4324}{5525}.$$  

Alternatively, every 3-card hand is equally likely, so we can use counting. There are $\binom{52}{3}$ total 3-card hands, and $\binom{48}{3}$ hands with only non-queen cards, which gives us the same result.

$$P(X = 0) = \frac{\binom{48}{3}}{\binom{52}{3}} = \frac{4324}{5525}.$$

• We will continue to use counting. The number of hands with exactly one queen amounts to the number of ways to choose 1 queen out of 4, and 2 non-queens out of 48.

$$P(X = 1) = \frac{\binom{4}{1} \binom{48}{2}}{\binom{52}{3}} = \frac{1128}{5525}.$$

• Choose 2 queens out of 4, and 1 non-queen out of 48.

$$P(X = 2) = \frac{\binom{4}{2} \binom{48}{1}}{\binom{52}{3}} = \frac{72}{5525}.$$

• Choose 3 queens out of 4.

$$P(X = 3) = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525}.$$

(b) We check:

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = \frac{4324 + 1128 + 72 + 1}{5525} = 1.$$

(c) From the definition, $E(X) = \sum_{k=0}^{3} kP(X = k)$, so

$$E(X) = 0 \cdot \frac{4324}{5525} + 1 \cdot \frac{1128}{5525} + 2 \cdot \frac{72}{5525} + 3 \cdot \frac{1}{5525} = \frac{3}{13}.$$

(d) No, they are not independent. As an example:

$$P(X_1 = 1)P(X_2 = 1) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}.$$

However,

$$P(X_1 = 1, X_2 = 1) = P(\text{the first and second cards are both queens}) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}.$$
3 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game $A$ 10 times and game $B$ 20 times. Each time you play game $A$, you win with probability $\frac{1}{3}$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game $B$ is similar, but you win with probability $\frac{1}{5}$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence “book” appears?

Solution:

(a) Let $A_i$ be the indicator you win the $i$th time you play game $A$ and $B_i$ be the same for game $B$. The expected value of $A_i$ and $B_i$ are

$$\mathbb{E}[A_i] = 1 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} = \frac{1}{3},$$

$$\mathbb{E}[B_i] = 1 \cdot \frac{1}{5} + 0 \cdot \frac{4}{5} = \frac{1}{5}.$$

Let $T_A$ be the random variable for the number of tickets you win in game $A$, and $T_B$ be the number of tickets you win in game $B$.

$$\mathbb{E}[T_A + T_B] = 3\mathbb{E}[A_1] + \cdots + 3\mathbb{E}[A_{10}] + 4\mathbb{E}[B_1] + \cdots + 4\mathbb{E}[B_{20}]$$

$$= 10\left(3 \cdot \frac{1}{3}\right) + 20\left(4 \cdot \frac{1}{5}\right) = 26$$

(b) There are $1,000,000 - 4 + 1 = 999,997$ places where “book” can appear, each with a (non-independent) probability of $\frac{1}{26^4}$ of happening. If $A$ is the random variable that tells how many times “book” appears, and $A_i$ is the indicator variable that is 1 if “book” appears starting at the $i$th letter, then

$$\mathbb{E}[A] = \mathbb{E}[A_1 + \cdots + A_{999,997}]$$

$$= \mathbb{E}[A_1] + \cdots + \mathbb{E}[A_{999,997}]$$

$$= \frac{999,997}{26^4} \approx 2.19.$$