1 Bayesian Darts

You play a game of darts with your friend. You are better than he is, and the distances of your darts to the center of the target are i.i.d. $U[0,1]$ whereas his are i.i.d. $U[0,2]$. To make the game fair, you agree that you will throw one dart and he will throw two darts. The dart closest to the center wins the game. What is the probability that you will win? *Note*: The distances *from the center of the board* are uniform.

2 Normal Distribution

Recall the following facts about the normal distribution: if $X \sim \mathcal{N}(\mu, \sigma^2)$, then the random variable $Z = (X - \mu)/\sigma$ is standard normal, i.e. $Z \sim \mathcal{N}(0,1)$. There is no closed-form expression for the CDF of the standard normal distribution, so we define $\Phi(z) = \mathbb{P}[Z \leq z]$. You may express your answers in terms of $\Phi(z)$.

The average jump of a certain frog is 3 inches. However, because of the wind, the frog does not always go exactly 3 inches. A zoologist tells you that the distance the frog travels is normally distributed with mean 3 and variance $1/4$.

(a) What is the probability that the frog jumps more than 4 inches?

(b) What is the probability that the distance the frog jumps is between 2 and 4 inches?

3 Sum of Independent Gaussians

In this question, we will introduce an important property of the Gaussian distribution: the sum of independent Gaussians is also a Gaussian.

Let X and Y be independent standard Gaussian random variables. Recall that the density of the standard Gaussian is

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

(a) What is the joint density of X and Y?

(b) Observe that the joint density of X and Y, $f_{X,Y}(x,y)$, only depends on the quantity x^2+y^2, which is the distance from the origin. In other words, the Gaussian is *rotationally symmetric*. Next, we will try to find the density of $X+Y$. To do this, draw a picture of the Cartesian plane and draw the region $x+y \le c$, where c is a real number of your choice.

(c) Now, rotate your picture clockwise by $\pi/4$ so that the line $X+Y=c$ is now vertical. Redraw your figure. Let X' and Y' denote the random variables which correspond to the $\pi/4$ clockwise rotation of (X,Y) and express the new shaded region in terms of X' and Y'.

(d) By rotational symmetry of the Gaussian, (X',Y') has the same distribution as (X,Y). Argue that $X+Y$ has the same distribution as $\sqrt{2}Z$, where Z is a standard Gaussian. This proves the following important fact: *the sum of independent Gaussians is also a Gaussian*. Notice that $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(0,1)$ and $Z \sim \mathcal{N}(0,2)$. *In general, if X and Y are independent Gaussians, then $X+Y$ is a Gaussian with mean $\mu_X+\mu_Y$ and variance $\sigma_X^2+\sigma_Y^2$.*

(e) Recall the CLT:

> If $\{X_i\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 < \infty$, then:
>
> $$\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}} \xrightarrow{\text{in distribution}} \mathcal{N}(0,1).$$

Prove that the CLT holds for the special case when the X_i are i.i.d. $\mathcal{N}(0,1)$.