CS 70 Discrete Mathematics and Probability Theory
Fall 2020 DIS 6B

1 Count it

Let's get some practice with counting!

(a) How many sequences of 15 coin-flips are there that contain exactly 4 heads?

(b) An anagram of HALLOWEEN is any re-ordering of the letters of HALLOWEEN, i.e., any string made up of the letters H, A, L, L, O, W, E, E, N in any order. The anagram does not have to be an English word.
How many different anagrams of HALLOWEEN are there?

(c) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each y must be a non-negative integer?

(d) How many solutions does $y_0 + y_1 = n$ have, if each y must be a positive integer?

(e) How many solutions does $y_0 + y_1 + \cdots + y_k = n$ have, if each y must be a positive integer?

Solution:

(a) This is just the number of ways to choose 4 positions out of 15 positions to place the heads, and so is $\binom{15}{4}$.

(b) In this 9 letter word, the letters L and E are each repeated 2 times while the other letters appear once. Hence, the number 9! overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of 2! for the number of ways of permuting the 2 L's among themselves and another factor of 2! for the number of ways of permuting the 2 E's among themselves). Hence, there are $9!/(2!)^2$ different anagrams.

(c) $\binom{n+k}{k}$. We can imagine this as a sequence of n ones and k plus signs: y_0 is the number of ones before the first plus, y_1 is the number of ones between the first and second plus, etc. We can now count the number of sequences using the "balls and bins" method (also known as "stars and bars").

(d) $n-1$. We can just enumerate the solutions here. y_0 can take values $1,2,\ldots,n-1$ and this uniquely fixes the value of y_1. So, we have $n-1$ ways to do this. But, this is just an example of the more general question below.

(e) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

2 Inclusion and exclusion

What is total number of positive numbers that smaller than 100 and coprime to 100?

Solution: It's enough to count the inverse: what is the total number of positive ingeters that smaller than 100 and not coprime to 100?

Not coprime to 100 means that the number either is a multiple of 2 or a multiple of 5. Then we have

49 numbers are multiple of 2. 19 numbers are multiple of 5. 9 numbers that are multiple of both 2 and 5.

So the total number is $49+19-9=59$, and there are 99 positive integers smaller than 100.

So in total, there are $99-59=40$ different number of positive numbers (smaller than 100) that are coprime to 100.

3 Identities

(a) $\sum_{i=0}^{n}(-1)^i\binom{n}{i}=0$

(b) $\sum_{i=0}^{n}\binom{r+i}{i}=\binom{r+n+1}{n}$

(c) $\sum_{i=0}^{n}\binom{r}{i}\binom{s}{n-i}=\binom{r+s}{n}$ (Note: Assuming $r>n, s>n$)

Solution:

(a) $0=(1-1)^n=\sum_{i=0}^{n}(-1)^i 1^{n-1}\binom{n}{i}$

(b) RHS $\binom{r+n+1}{n}$ can be viewed as counting the number of subsets of $\{1,2,...,n+r+1\}$ of size n. $\sum_{i=1}^{n}\binom{r+i}{i}$ can be viewed as counting the same thing but in a different way. It first specifies the smallest element that is 'NOT' in the selected subset. For example if 1 is NOT in the subset then there are $\binom{n+r+1-1}{n}$ ways of different subsets. If 2 is the smallest that not in the subset then 1 is in the subset and there are $\binom{n+r+1-2}{n-1}$ different ways remaining, etc. down to if $n+1$ is the smallest number that not in the set, then $1,2,3,...,n$ are in the subset then we only have $\binom{n+r+1-(n+1)}{0}$ different ways. So RHS and LHS counts the same thing.

(c) RHS counts the total number of different subsets from $\{1,2,3,4,...,r,r+1,...,r+s\}$ that has size n, LHS counts the same thing by specify how many elements is selected from $\{1,2,3,4,...,r\}$ and how many of them are selected from $\{r+1,r+2,...,r+s\}$. If i of them are selected from the first set, then $n-i$ of them must be selected from the second set, the total number is $\binom{r}{i}\binom{s}{n-i}$. And we iterate through all possible i.

4 Largest binom

For which value(s) of k is $\binom{n}{k}$ maximum? Prove your answer.

Solution: When n is odd, $\binom{n}{\frac{n+1}{2}}$ and $\binom{n}{\frac{n+1}{2}-1}$ are maximum. When n is even, $\binom{n}{n/2}$ is maximum.

To prove this, we need the following equality:

$$\binom{n}{k} = \frac{n-k+1}{k}\binom{n}{k-1}$$

Proof: $\frac{n-k+1}{k}\binom{n}{k-1} = \frac{n!}{(k-1)!(n-k+1)!}\frac{n-k+1}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$

We noticed that $\frac{n-k+1}{k} \geq 1$ when $k \leq \frac{n+1}{2}$ for n is odd case, and $\frac{n}{2}$ for n is even case.

So $\binom{n}{k} \geq \binom{n}{k-1}$ when $k \leq \frac{n+1}{2}$ for n is odd case, and $\frac{n}{2}$ for n is even case.