CS 70 Discrete Mathematics and Probability Theory
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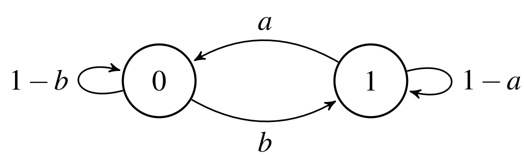
1 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains.

1. (Irreducibility) A Markov chain is irreducible if, starting from any state i, the chain can transition to any other state j, possibly in multiple steps.
2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i,i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathscr{X}$. If $d(i) = 1\ \forall i \in \mathscr{X}$, then the Markov chain is aperiodic; otherwise it is periodic.
3. (Matrix Representation) Define the transition probability matrix P by filling entry (i, j) with probability $P(i, j)$.
4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

(d) Construct a transition probability matrix using the above Markov chain.

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

2 Aperiodicity

(a) Can you find a finite irreducible aperiodic Markov chain whose distribution does not converge?

(b) Construct a finite Markov chain that is a sequence of i.i.d. random variables. Is it necessarily irreducible and aperiodic? What is its invariant distribution?

3 Markov Maze

Tom is stuck on a 3×3 grid as shown below. At every minute, he looks at all possible directions in which he could go (out of up, down, left, right), randomly chooses one with equal probability, and moves one square in that direction. Let's study Tom's behavior.

First, we define a convenient state space. Let the set of states be $K = \{1, 2, 3\}$, where state 1 represents being in a corner square, state 2 represents being in an edge square, and state 3 represents being in the center square. We could have defined one state for every square in the grid, but that would be unnecessarily tedious; these states take advantage of the symmetry of the grid.

(a) Write down the transition matrix and the balance equations for this problem.

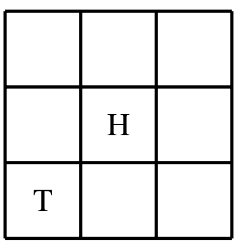
(b) Is this Markov chain irreducible?

(c) Is this Markov chain periodic? If so, what is its period?

(d) For each square in the grid, calculate the long-term fraction of time Tom will spend at it.

(e) Do the values of $\mathbb{P}[X_n = i]$ necessarily converge as n increases? If so, what values do they converge to? If not, give a counterexample where the values of $\mathbb{P}[X_n = i]$ oscillate.

(f) Now assume Tom starts from the lower-left square of the grid, and he wants to get back to his house, which is in the center square of the grid. Compute the expected time before Tom gets back to his house.



4 Limiting Distribution

[*This problem is optional and may not be covered during discussion.*]

(a) What is $\lim_{n\to\infty} n^{-1} \sum_{m=0}^{n} \mathbb{1}\{X_m = i\}$?

(b) True/False: $n^{-1} \sum_{m=0}^{n} \mathbb{1}\{X_m = i\} \xrightarrow[n\to\infty]{} \pi(i)$, where π is the invariant distribution, but only if the Markov chain is irreducible and aperiodic.

(c) True/False: An irreducible Markov chain will always converge to its invariant distribution.

(d) Construct a Markov chain that is not irreducible but that has a unique distribution and is such that its distribution converges to that unique invariant distribution, for any initial distribution.

(e) Show a Markov chain whose distribution converges to a limit that depends on the initial distribution.